REPRINTED FROM:

THE COLLECTED PAPERS OF

Albert Einstein

VOLUME 3

THE SWISS YEARS:
WRITINGS, 1909–1911

Martin J. Klein, A. J. Kox, Jürgen Renn, and Robert Schulmann
EDITORS

Jed Buchwald, Jean Eisenstaedt, Don Howard, John Norton, and Tilman Sauer
CONTRIBUTING EDITORS

Ann Lehar, Rita Lübke, Annette Pringle, and Shawn Smith
EDITORIAL ASSISTANTS

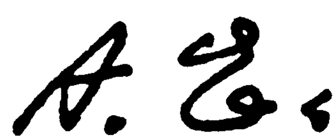
ENGLISH TRANSLATION OF SELECTED TEXTS

Anna Beck, TRANSLATOR
Don Howard, CONSULTANT

DOC. 11

Lecture Notes for Course on Electricity and Magnetism at the University of Zurich, Winter Semester 1910–1911

(pp. 251–327 in translation volume)



Princeton University Press
1993

Doc. 10

Comments on P. Hertz's Papers: "On the Mechanical Foundations of Thermodynamics"[1]

by A. Einstein.

[*Annalen der Physik* 34 (1911): 175–176]

In his superb papers titled as above, Mr. Hertz has criticized two passages in my papers
[3] on the same topic. In the following, I will briefly comment on these criticisms, noting
that what is said here is the result of an oral discussion with Mr. Hertz, in which we came
to a perfect agreement regarding both points in question.

1. In the penultimate section of §13 of his second paper, Hertz criticizes a derivation
[4] that I gave of the entropy law for irreversible processes. I consider this criticism totally
valid. I was not satisfied with my derivation even then, which is why I soon thereafter
[5] produced a second derivation, also cited by Mr. Hertz.

2. The comments contained in §4 of his first paper that are directed against an
argument about thermal equilibrium contained in my first paper in question[2] are based
on a misunderstanding caused by an all-too terse and insufficiently careful formulation
[6] of that argument.

However, since the topic has been adequately elucidated in works by other authors,
and since, moreover, a detailed discussion of this specific point is not likely to claim much
interest, I do not wish to elaborate on it here. I only wish to add that the road taken by
Gibbs in his book, which consists in one's starting directly from the canonical ensemble,
[7] is in my opinion preferable to the road I took. Had I been familiar with Gibbs's book
[8] at that time, I would not have published those papers at all, but would have limited
myself to the discussion of just a few points.

Zurich, October 1910. (Received on 30 November 1910)

[1] [1] A. Einstein, *Ann. d. Phys.* 9 (1902): 425 and 11 (1903): 176.
[2] [2] P. Hertz, *Ann. d. Phys.* 33 (1910): 225 and 537.

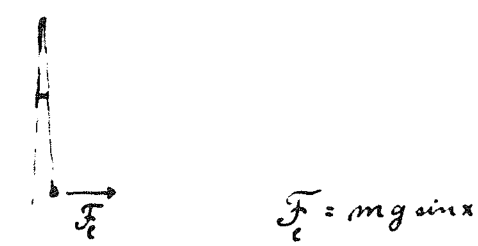
Doc. 11

Lecture Notes for Course on Electricity and Magnetism at the University of Zurich, Winter Semester 1910/11[1]

Electrostatics [p. 1]

If one rubs glass, sealing wax, or other bodies with other bodies, then after this procedure they will (temporarily) exert forces on each other that were not observable before, without their having been otherwise influenced in a perceptible manner. One says that they are "electrified," where by this word one does not denote anything but what has been said. Metals & many other bodies can be electrified only if affixed to a prop of glass or sealing wax etc., or suspended by a silk thread. A body can be electrified not only by rubbing but also by bringing it into contact with an electrified body.

Let us examine the laws according to which electrified bodies act upon one another, assuming for the sake of simplicity that the bodies are small compared with the distances between them. The forces exerted by these bodies on each other act in the direction of the connecting lines (equality of action & reaction, we can measure them absolutely by the methods of mechanics, for example in the following way:



Consider now many bodies, say small metal balls suspended by silk threads, and let [p. 2]
us suppose that we have determined that the forces that *any two of them* exert on each other, and assume, for the time being, that they are at a distance R that always stays the same. We designate attractive forces as negative, repulsive as positive.

If we combine the bodies 1 2 3 .. with the body a of our group, we obtain the forces F_{1a}, F_{2a}, F_{3a} .... .. If we combine the same bodies 1 2 3 .. with the body b, we obtain the forces

$$F_{1b}\ F_{2b}\ F_{3b}$$

Experience shows that $F_{1a} : F_{2a} : F_{3a} .. = F_{1b} : F_{2b} : F_{3b} ..$ Thus, the effects of the bodies 1 2 3 .. another body always stand in the same ratio no matter how that other body has been chosen. Hence we can characterize the electrical influence of *one* el. body by means of a number, if we have assigned an arbitrarily chosen number, for example the number 1, to the influence of one of the bodies.[2] This number is called the quantity of electricity. It follows from this definition that the force f exerted by two bodies on each other is directly proportional to their quantities of electricity.

$$F = k \cdot e_1 e_2.$$

However, k also depends on the distance.

Further, it follows from experiments that this force is inversely proportional to the square of the distance, so that we have, with another interpretation of the constant k,

$$F = k\frac{e_1 e_2}{r^2},$$

where k no longer depends on the distance but only on our choice of the body in our
[p. 3] group to which we have assigned the quantity of electricity 1.

The sign of k is determined by our earlier stipulation in conjunction with experience. That is to say, it has been found that quantities of electricity that are alike according to the above definition repel each other. Thus, k is a positive constant. Its value depends on what we stipulate as the unit of the quantity of electricity. However, we may also freely choose k and thereby define the unit of the quantity of electricity. We do that by setting $k = 1$. We have then

$$F = \frac{e_1 e_2}{r^2}$$

In order to measure a quantity of electricity absolutely after according to this <kind of> definition,[3] one has to measure, in principle, a force and a length, which quantities occur in the form

$$e = \sqrt{\text{force}} \cdot \text{length} = M^{1/2}L^{3/2}T^{-1}$$

This is the "dimension" of the electrostatically measured quantity of electricity.

We must mention a few more facts that are of fundamental importance for the foundations of the theory.

If a quantity of electricity e_a is subjected to the action of two quantities of electricity
[p. 4] e_1 & e_2, one finds the force acting on e_a from the law of the parallelogram of forces. In the special case where e_1 & e_2 are very close to each other, their effects on e_a will add up

algebraically; in other words: the quantity of electricity of a system of bodies is equal to the sum of the quantities of electricity of the system's individual bodies.

This principle can be further extended, given the character of our experience with electrified bodies. If bodies with quantities of electricity e_1 & e_2 are brought into contact with one another, then, in general, their electric state will change. But their action at a distance on a third e. q. e_a will not change upon the contact, and so the sum of the electrical quantities will not change either. (Important law of the constancy of the sum of quantities of electricity, an exception to which has never been found.)

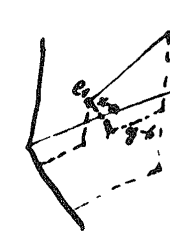
We endow these two laws with a tangible, physical meaning by imagining that the substrate for the quantity of electricity is some sort of indestructible matter, which, however, must be thought of as being present in a positive and a negative modification, because the experiments alluded to above show the existence of positive as well as negative electrical quantities (in the case of attractive forces).

One more thing has to be added to complete what has been said so far, for there is
no way to decide which sign to ascribe to a specific given electrical quantity, because the [p. 5]
interaction between two e. q. only makes it possible to decide whether the two have to be assigned *like* or *opposite* signs. But all that is needed, therefore, is to fix the sign in a specific case (glass rubbed with wool is positive), in order to fix signs for all other quantities of electricity.

In completing what has been said about the auxiliary representation of positive and negative electricity, it should be added that one imagines that the interactive forces act between the electricities and are transferred from them to the carriers of electricity (bodies) to which they are bound. We further complete the picture by the assumption that not only the algebraic sum of the electrical quantities, but also the sum of the electricities of each of the signs is constant—a proposition that is part of the picture and that cannot be either directly confirmed or directly discomfirmed by experiment.

The action of a system of electric masses $(e_1\ e_2\ \ldots.)$ on a pointlike quantity of electricity (e).

An electrical quantity $e_1(x\ y\ z)$ exerts the force K on a quantity of electricity $e(a, b, c)$.[4] We have



$$K = \frac{e_1 e}{r^2}, \quad \text{where} \quad r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

The direction cosines of this force are $\frac{x-a}{r}, \frac{y-b}{r}, \frac{z-c}{r},$

so that its components are

[p. 6]

$$\left.\begin{aligned} K_{x1} &= \frac{e_1 e}{r_1^2}\,\frac{x-a_1}{r_1} \\ K_{y1} &= \frac{e_1 e}{r_1^2}\,\frac{y-b_1}{r_1} \\ K_{z1} &= \frac{e_1 e}{r_1^2}\,\frac{z-c_1}{r_1} \end{aligned}\right\} \quad (1)$$

If several masses e_1 e_2 .... act simultaneously on mass e, we get[5]

$$K_x = \Sigma K_{x1} = \frac{e_1 e}{r_1^2}\,\frac{x-a_1}{r_1} + \frac{e_2 e}{r_2^2}\,\frac{x-a_2}{r_2}\ldots\ldots = e\cdot\sum_1^n \frac{e_1}{r_1^2}\,\frac{x-a_1}{r_1}$$

— — — — — — — — — — — — — — — — — —

— — — — — — — — — — — — — — — — — —

For a given distribution of the masses e_1 etc., and a given position for e, these force components are proportional to the e.q. e. But the sums appearing on the right-hand side depend only on e_1 e_2 ... & the test point. These sums

$$\sum_1^n \frac{e_1}{r_1^2}\,\frac{x-a_1}{r_1} = X \quad \text{(other components } Y\ Z\text{)}$$

are called the X-component of the electric force or field strength. It is equal to the force exerted on the unit of electricity. $X\,Y\,Z$ is a vector which is related to the vector of the force acting upon the e quantity e in the following way:

$$K_x = eX \quad K_y = eY \quad K_z = eZ\ldots\ldots(2)$$

If one draws from every spatial point a directed straight line in the direction of the field intensity, one gets a picture of the course of the field intensity, of the vector field $X\,Y\,Z$
[p. 7] that brings about the (possible) actions of forces deriving from the quantities e_1 e_2 etc. This field is determined chiefly by 3 spatial functions (X Y and Z). However, these can be reduced to a single spatial function. For we have

$$X = \sum \frac{e_1}{r_1^2}\,\frac{x-a_1}{r_1} = \sum \frac{e_1}{r_1^2}\,\frac{\partial r_1}{\partial x} = -\frac{\partial}{\partial x}\left\{\sum\left(\frac{e_1}{r}\right)\right\},$$

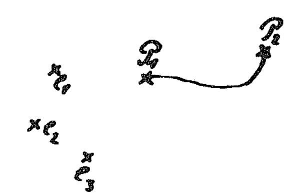
since because $r_1^2 = (x-a_1)^2 + \cdot + \cdot$, $\quad r_1 dr_1 = (x-a_1)dx + \cdot + \cdot$ Hence, if we set $\sum \frac{e_1}{r_1} = \varphi$, we get

$$\left.\begin{aligned} X &= -\frac{\partial\varphi}{\partial x} \\ Y &= -\frac{\partial\varphi}{\partial y} \qquad \varphi = \sum\frac{e_1}{r_1} \\ Z &= -\frac{\partial\varphi}{\partial z} \end{aligned}\right\} \quad (3)$$

Thus, $X\,Y\,Z$ can be described as derivatives of *one* spatial function φ. We call φ the potential of the masses in question.

The Physical Meaning of the Potential

We consider the electrical unit mass in the field of the e. q. e_1, e_2, e_3.... We move the unit m. from the point P_1 to the point P_2. For an infinitesimally small portion of the path with projections $dx\,dy\,dz$, the work performed by the forces of electric origin equals $Xdx + Ydy + ZdZ$. The total work is therefore $A = \int_{P_1}^{P_2} Xdx + Ydy + Zdz$



With the help of (3), this work can be given the form

$$A = -\int \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = -\int d\varphi,$$

where $d\varphi$ denotes the total change of φ when the element $dx\,dy\,dz$ is traversed. Hence we obtain

$$A = \varphi_1 - \varphi_2 \ldots. \ (4)$$

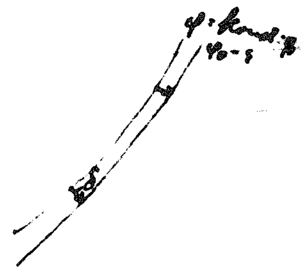
Thus, the work done on the unit electr. mass between two points is equal to the potential [p. 8]
drop between these two points φ is independent of the choice of the coordinate system. This quantity is totally independent of the shape of the path. Hence, if the unit pole

describes a closed curve, i.e., if P_1 & P_2 coincide, $\varphi_1 = \varphi_2$, and hence the work $A = 0$. This fact contains the more profound interpretation of the reason why the vector XYZ of the el. field strength is derivable from a potential. If the integral were not to vanish for a closed curve, it would be possible to produce work from nothing, without limit, by means of electrical quantities.

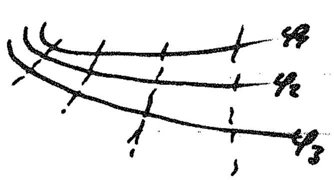
The Theorems of Laplace and Gauss. Lines of Force

<Here give a little kiss to his poor!>[6]

The funct φ provides a graphic overview of the course of the el. field. If one thinks of a surface φ = const., then the field vector XYZ will be perpendicular to the surface φ = const. Because every derivative $-\frac{\partial\varphi}{\partial s}$ taken in the direction of a line element in the surface vanishes. If we think of two adjacent surfaces $\varphi = \varphi_0$ & $\varphi = \varphi - \varepsilon$, we



will have $-\frac{\partial\varphi}{\partial n} = \frac{\varepsilon}{\delta}$, and since ε is everywhere constant along the two surfaces, $\frac{1}{\delta}$ is a relative measure for $-\frac{\partial\varphi}{\partial n}$, i.e., for the el. field strength, or—as we will call it in brief for the el. force. An additional aid for intuitive visual representation is provided by the concept of lines of force, i.e., of lines that at each point have the same direction as the electric force. According to what we have said, these lines of force everywhere intersect the surfaces of equal potential perpendicularly. Beyond this, we will see that the density of these lines of force is proportional to the field intensity. But in order to do



[p. 9] this, we must first derive a few laws.

The Theorems of Laplace & Gauss

If only one charge is pres., then $\varphi = \frac{e}{r}$, where

$$r = +\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}\,.$$

Diff, we obt.

$$\frac{\partial\varphi}{\partial x} = -\frac{e}{r^2}\frac{x-a}{r} = -\frac{e}{r^3}(x-a)$$

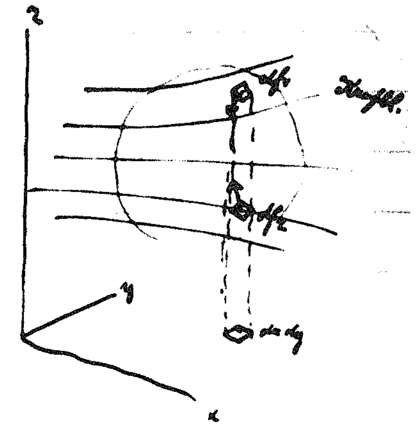
$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{e}{r^3} + \frac{3e}{r^4}\frac{(x-a)^2}{r}$$

From this, $\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \ldots.$ (5) also holds for an arbitrary number of masses (Laplace's theorem)

We can express this theorem in still another form if we use the field intensity instead of the derivatives of φ.

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0 \ldots. (5a)$$

We can give this theorem a new form by integrating over a volume bounded by a closed surface that contains no electric masses.



$$\int\frac{\partial Z}{\partial z}dx\,dy\,dz$$

Portion of an element

$$dx\,dy = dx\,dy\int\frac{\partial Z}{\partial z}\,dz = dx\,dy\,(Z_1 - Z_2)$$

if n_1 and n_2 are the inwardly oriented normals, then[7]

$$dx\,dy = -\,df_1 \cos(n_1 z) = df_2 \cos(n_2 z)$$

We can set $-\Sigma Z \cos nz\, df$ over the two elements Every other element $dx\,dy$ has the same form, so that when one finally replaces the sum with the integral, one obtains[8] [p. 10]

$$\int\frac{\partial Z}{\partial z}d\tau = -\int Z \cos nz\, ds$$

Applying this theorem three times, one obtains

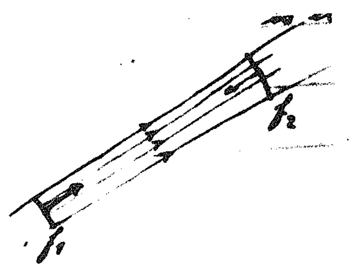
$$0 = \int\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)d\tau = -\int(X \cos nx + Y \cos ny + Z \cos nz)\,ds\,.$$

Considering that the expression in the brackets is in fact the field component N in the direction of the inward normals, we obtain

$$\int N ds = 0. \quad (6)$$

which we can also write in the form $\int \frac{\partial \phi}{\partial n} ds = 0$

This theorem yields us a further property of the field of electric lines of force. Define tube of force & write down above theorem for it. Integral vanishes on the surface. On the initial and terminal cross section we have

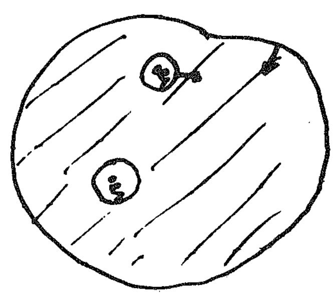


$$N_1 f_1 = N_2 f_2.$$

This vanishes. $\frac{N_2}{N_1} = \frac{f_1}{f_2}$. Thus, the field strengths vary inversely as the surfaces of the tubes of force. If one draws a number of lines of force through f_1 and continues them up to f_2, then the density of these lines of force will likewise be inversely proportional to the surface areas, and thus directly proportional to the field intensities N. Thus, one can draw unending lines of force in the field, so that line density = field strength. This is why the lines of force afford a quite complete & direct intuitive visual representation of a field.

[p. 11] Equation (6) expresses the one special case of the so-called Gauss's theorem.[9] This equation can easily be extended to the case where the closed surface encloses electric masses $e_1\, e_2 \ldots.$

We extend the surface integral to the volume bounded by the giv. surface F and the auxiliary spherical surf. $K_1\, K_2$ etc.



$$\int_F N ds + \int_{K_1} N ds + \int_{K_2} N ds \ldots = 0.$$

We seek integral extended over sphere K_1 We divide the total field into 1. $X_1\, Y_1\, Z_1\, N_1$, which derives from e_1, & second, the rest $X'Y'Z'N'$

The surface integral $\int_{K_1} N' ds$ vanishes, $\int_{K_1} N_1 ds = \frac{e_1}{r^2} 4\pi r^2 - 4\pi e_1$ We thus obtain

$$\int N ds = -4\pi \sum e_1 \quad \text{(General form of Gauss's theorem.)}$$

Continuously Distributed Electricity

So far we have assumed that electricity is unalterably bound to small bodies (treated as points). But the character of experience favors the assumption that electricity is spatially distributed. We must generalize our investigations in this sense. To begin with, we think of electricity as continuously distributed, $\rho d\tau$ being the quantity of electricity in the space element $d\tau$. ρ is the difference between the densities of positive and negative electricity [p. 12] at one locus, as we imagine it. We assume that the electricities are movable relative to ponderable matter, and that they cannot undergo any other changes except those of position. This model is suggested by the earlier-mentioned empirical law of the constancy of the quantity of electricity in the electrical balance between two small bodies.

The following should be noted here. We have seen how experience led to the introd. of the concept of the quantity of electricity. it was defined by means of the forces that small electrified bodies exert on each other. But now we extend the application of the concept to cases in which this definition cannot be applied directly as soon as we conceive the el. forces as forces exerted *on electricity* rather than on material particles. We set up a conceptual system the individual parts of which do not correspond directly to empirical facts. Only a certain totality of theoretical material corresponds again to a certain totality of experimental facts.[10]

We find that such an el. continuum is always applicable only for the representation of el. states of affairs in the interior of ponderable bodies. Here too we define the vector [p. 13] of el. field strength as the vector of the mech. force exerted on the unit of pos. electr. quantity inside a body. But the force so defined is no longer directly accessible to exp. It is one part of a theoretical construction that can be correct or false, i.e., consistent or not consistent with experience, only *as a whole*. The laws that we found empirically for small electrified bodies we now apply to the fictional electricity itself.

We invest. the pot. of cont. distribution

$\varphi = \int \rho \frac{d\tau}{r}$ R small radius sphere about the test point region decomposed polar coordinates introduced

$$c - z = r \cos \vartheta$$
$$a - x = r \sin \vartheta \cos \omega \quad \text{volume el} \cdot r^2 \sin\vartheta\, dr d\omega d\vartheta$$
$$b - y = r \sin \vartheta \sin \omega$$

In small sphere $\int_K \frac{\rho d\tau}{r}$ replaceable by $\int \rho_0 r \sin\vartheta\, dr d\omega d\vartheta$ always finite. Thus, the integral is not infinite.

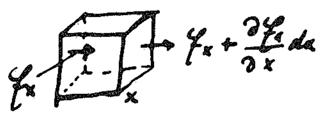
$$\frac{\partial\varphi}{\partial z} = \int \frac{\rho\, d\tau}{r^2}\frac{c-z}{r} = \int_R + \int_K \rho \cos\vartheta \sin\vartheta \; dr\; d\omega\, d\vartheta .$$

The second int. is finite.[11] Hence field strength always finite. One proves that when ρ with all derivatives is continuous, the same must be true of φ.

The equation $\Delta\varphi = 0$ is not valid here. We find the corresponding theorem by
[p. 14] applying Gauss's theorem to an arb. closed surface inside the continuum.

$$\boxed{\int \mathfrak{E}_n d\sigma = -\int 4\pi\rho d\tau ,}$$

where $\mathfrak{E}_n$ denotes the component of the el. field strength along the inward normal. First we apply the theorem to the special case where the surface is the boundary of an elementary parallelepiped. The right side becomes $-4\pi\rho\, d\tau$. The left side



$$\mathfrak{E}_x dy\, dz + -\left(\mathfrak{E}_x + \frac{\partial \mathfrak{E}_x}{\partial x}dx\right) dy\, dz$$

— — — — — — — — — —

— — — — — — — — — —

or $$-\left(\frac{\partial \mathfrak{E}_x}{\partial x} + \frac{\partial \mathfrak{E}_y}{\partial y} + \frac{\partial \mathfrak{E}_z}{\partial z}\right) d\tau$$

If the two sides are set equal, one obtains

$$\boxed{\frac{\partial \mathfrak{E}_x}{\partial x} + \frac{\partial \mathfrak{E}_y}{\partial y} + \frac{\partial \mathfrak{E}_z}{\partial z} = 4\pi\rho}$$

If one replaces $\mathfrak{E}_x$ etc. by the derivatives of the potential, one obtains

$$\boxed{+\Delta\varphi = -4\pi\rho}$$

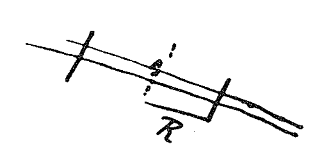
This is Poisson's theorem.

Distribution of the Electricity on Conductors

A conductor is a substance in which the electricity is freely movable. Equilibrium possible only if no forces act on el. in the interior. $\mathfrak{E}_x$ etc. vanish. Poisson's theorem applied to a point in the interior of the conductor yields $\rho = 0$. Thus, the electric masses sit only on the surface, & in the interior of the conductor φ = const.

[p. 15] Since the electricity is distributed two-dimensionally on the surface, we must consider a two-dimensionally distributed potential.

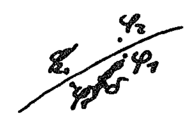
1) Potential is uniform over surface. A little piece of the surface is cut out by a cylinder around the spot under investigation. That which derives from the external part of the covering is uniform. That which derives from the internal part vanishes for small radius; for[12]



$$\varphi_i = \int \frac{\eta\, d\sigma}{r} = \eta_0 \int_0^R \frac{2\pi r\, dr}{r} = 2\pi R,$$

which decreases with decreasing R.
From the constancy of φ it follows that the tangential components of $\mathfrak{E}$ on the two sides of the layer are equal.

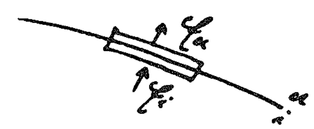
$$\left.\begin{aligned}\varphi_1 &= \varphi_2\\ \varphi_1' &= \varphi_2'\end{aligned}\right\}\varphi_{1'} - \varphi_1' = \varphi_2 - \varphi_2'$$



$$\frac{\varphi_1 - \varphi_1'}{\delta} = \frac{\varphi_2 - \varphi_2'}{\delta} \quad \text{or} \quad \boxed{\mathfrak{E}_{t2} = \mathfrak{E}_{t1}}$$

From this it follows $\mathfrak{E}_t$ vanishes on the external surface of a conductor, i.e., that the lines of force must intersect the surface of the conductor perpendicularly.
2) How does the normal component behave on the two sides?
This follows at once from Gauss's theorem.[13]

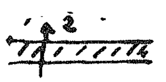
$$4\pi\sigma\, df = \mathfrak{E}_i df - \mathfrak{E}_a df$$



or

$$\mathfrak{E}_{ni} - E_{na} = -4\pi\sigma. \quad \text{special case} \quad \mathfrak{E}_{ni} = 0 \quad \mathfrak{E}_{na} = 4\pi\sigma$$

or $\left(\frac{\partial\varphi}{\partial n}\right)_i - \left(\frac{\partial\varphi}{\partial n}\right)_a = 4\pi\sigma$, if both normals are taken toward the *external* side.
Force on piece of the conductor surf. [p. 16]



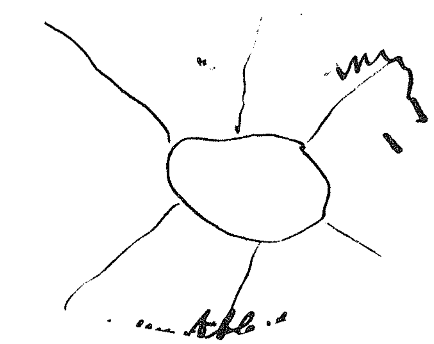
$$\int \frac{\partial \mathfrak{E}_2}{\partial z} dz = \mathfrak{E}_{2a} - \mathfrak{E}_{2i} = 4\pi \int \rho\, dz = 4\pi\sigma$$

$$+\,4\pi\rho = \frac{\partial \mathfrak{E}_2}{\partial z}.$$

$$4\pi \int \rho \mathfrak{E}_2\, dz = \int \mathfrak{E}_2 \frac{\partial \mathfrak{E}_2}{\partial z} dz = \frac{1}{2}(\mathfrak{E}_{2a}^2 - \mathfrak{E}_{2i}^2) \Big| \mathfrak{K}\mathrm{raft} = \frac{1}{8\pi}\mathfrak{E}_2^2$$

The problem of finding the distribution of electricity on a conductor is now easy to formulate mathematically if we further stipulate that the potential should be constant at ∞. If all effective el. masses are at a finite distance, its value there is zero. For φ can be determined from the following conditions:

1) φ = const = P_0 inside the body
2) $\Delta\varphi = 0$ outside the body.
3) φ constant on the surface of the body. φ together with the derivatives in the external region.
4) φ vanishes at ∞.



We prove later that these conditions are *sufficient*.

~~For the~~ difference φ_1 of two solutions φ_1 must ~~vanish~~ ~~*outside*~~ on the surface. <Thus, if there existed a ~~closed surface~~ anywhere in the external region>

We now choose a closed surface in the external region

$$\int\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 = -\int \varphi\frac{\partial\varphi}{\partial n}ds$$

If φ is determined in accordance with these conditions, one obtains the surface density η by means of the relation $4\pi\eta = \mathfrak{E}_n = -\frac{\partial\varphi}{\partial n}$, where the normal is directed toward the outer side of the conductor. One obtains the total charge by integrating η over the surface.

[p. 17] Example. Let the given body be a sphere. We show that the solution $\varphi = \frac{\alpha}{r}$ in the external region and $\varphi = P$ in the internal region satisfies all the conditions.

1) satisfied

2) satisfied, because $\Delta\left(\frac{\alpha}{r}\right) = 0$

3) satisfied, if $\frac{\alpha}{R} = P$

4) satisfied.

We determine the charge e.

$$e = \int \eta\, d\sigma = \int \frac{\alpha}{4\pi R^2} R^2 d\kappa = \alpha \qquad \eta = -\frac{1}{4\pi}\left(\frac{\partial\varphi}{\partial r}\right)_R = +\frac{1}{4\pi}\frac{\alpha}{r^2}$$

Thus, we obtain

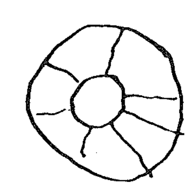
$$\boxed{\begin{array}{l}\varphi = \frac{e}{r} \\ e = RP\end{array}}$$

This shows that e is proportional to the potential difference P. This holds not only for a sphere but quite generally. For let the problem be solved for a specific P. One then finds the solution for a $P^x = \lambda_{\text{const}}P$ by using the function $\varphi^x = \lambda\varphi$ instead of φ. Thus, $\frac{e}{P}$ depends only on the shape of the conductor and is called the capacity of the latter. The capacity of the sphere is equal to its radius.

Instead of a single conductor, let us think of one surrounded by a conducting casing.

1) $\varphi = P_1$ in the interior $\varphi = P_2$ in casing
2) $\Delta\varphi = 0$ bet. body & casing
3) constancy req.

Then $\lambda P_1 \lambda P_2 \lambda\varphi$ solution λe el. quantity on body as well as on casing



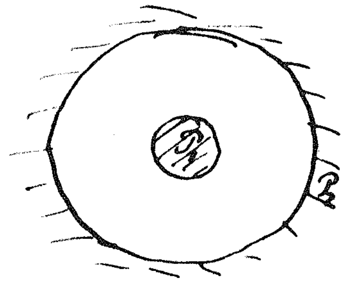
Charge dep. only on pot diff.

$$\frac{P_1 - P_2}{e} = \frac{\lambda P_1 - \lambda P_2}{\lambda e} = c \text{ capacity, (mutual)}$$

Example parallel plate condenser[14] [p. 18]

$$E = -\frac{\partial\varphi}{\partial x} = 4\pi\sigma = \frac{P}{\delta} \quad e = \sigma f = \frac{Pf}{4\pi\delta} \quad C = \frac{f}{4\pi\delta}$$

Example concentric hollow spheres.



$$\varphi = \frac{\alpha}{r} + \beta$$

$$\frac{\alpha}{R_1} + \beta = P_1 \qquad \alpha\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = P_1 - P_2$$

$$\frac{\alpha}{R_2} + \beta = P_2 \qquad \alpha = (P_1 - P_2)\frac{R_1R_2}{R_2 - R_1}$$

β does not interest us.

$$\varepsilon = \int \eta\, d\sigma = \frac{1}{4\pi}\int -\frac{\partial\varphi}{\partial r}d\sigma = \frac{1}{4\pi}\frac{\alpha}{R^2}4\pi R^2 = \alpha$$

$$\frac{\varepsilon}{P_1 - P_2} = \frac{R_1R_2}{R_2 - R_1} = \text{mutual capacity}$$

Even simpler derivation (center)

Example concentric cylinders, φ depends only on $\delta = \sqrt{x^2 + y^2}$. One could set $\Delta\varphi = 0$ for this special case & integrate. Even simpler, apply Gauss's theorem directly. e = el. charge per unit length

$$4\pi e = 2\pi r \mathfrak{E}_r$$

$$\mathfrak{E}_r = \frac{2e}{r} = -\frac{\partial\varphi}{\partial r}$$

$$\varphi = -2e \lg r + \text{const} = -2e \lg \frac{r}{c}$$

Boundary conditions yield

$$P_1 = -2e \lg \frac{R_1}{c}$$

$$P_2 = -2e \lg \frac{R_2}{c}$$

$$P_1 - P_2 = -2e \lg \frac{R_1}{R_2} = 2e \lg \frac{R_2}{R_1}$$

$$\text{Capacitance} = c = \frac{e}{P_1 - P_2} = \frac{1}{2 \lg \frac{R_2}{R_1}}$$

Becomes zero when $R_2 = \infty$. Only slightly dependent on the ratio $\cdot \frac{R_2}{K_1}$

Electrical reflection of two spheres. [p. 19]
Uniqueness of the solution. Green's theorem.

$$\int\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \cdot + \cdot\right)d\tau = -\int U\frac{\partial V}{\partial n}d\sigma - \int U\Delta V d\tau$$

$$\int dy dz \int \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}dx = \underbrace{\int dy dz}_{-\alpha\sigma\cos nx}\left(U\frac{\partial V}{\partial x}\right) - \int U\frac{\partial^2 U}{\partial x^2}d\tau\right.$$

$$-\int U\left(\frac{\partial V}{\partial x}\cos nx + \cdot + \cdot\right) = \int U\frac{\partial V}{\partial n}d\sigma$$

The above equation is a form of Green's theorem. If we set $U = V$ & $\Delta U = 0$ & on the surface $U = 0$, then $\int\left(\frac{\partial U^2}{\partial x} + \cdot + \cdot\right)d\tau = 0$ Provides the proof of uniqueness. It is easy to calc. U in a point if one knows U & $\frac{\partial U}{\partial n}$ on the boundary surface of a space.

Electrical Energy

We start again from system of small electrified bodies. First two bodies a & b. Mutual force $\frac{e_a \cdot e_b}{r^2} = F$

Components

$F\dfrac{x_b - x_a}{r}$	dx_b	$-F\dfrac{x_b - x_a}{r}$	dx_a
$F\dfrac{y_b - y_a}{r}$	dy_b	$-F\dfrac{y_b - y_a}{r}$	dy_a
$F\dfrac{z_b - z_a}{r}$	dz_b	$-F\dfrac{z_b - z_a}{r}$	dz_a

$$dA + F\left\{\frac{(x_b - x_a)(dx_b - dx_a) + \cdot + \cdot}{r}\right\} = F\frac{r\,dr}{r} = F\,dr$$

Can also be understood geometrically

But $F = -\dfrac{\partial \Phi_{ab}}{\partial r}$, where $\Phi_{ab} = \dfrac{e_a e_b}{r_{ab}}$.

$$dA = -\frac{\partial \Phi_{ab}}{\partial r_{ab}} dr_{ab} = -d\Phi_{ab}$$

If many masses are present, one obtains the analogous expression, but one has to sum over all combinations.

[p. 20] $$dA = -\sum\sum d\Phi_{ab} = -d\left\{\sum\sum \Phi_{ab}\right\} = -d\Phi \qquad \Phi = \sum\sum \frac{\varepsilon_a \varepsilon_b}{r_{ab}}.$$

The elementary work is equal to the decrease in the function Φ, which we may call the potential energy of the electric forces or simply potential energy. When doing the double sum, each combination should be counted *once*.

But if one proceeds by first comb. the mass 1 with all the other masses, then mass 2 with all the others etc., then one counts each combination twice; hence one has to set

$$\Phi = \frac{1}{2}\sum\sum \frac{e_a e_b}{r} \quad \text{or} \quad \varphi = \frac{1}{2}\sum e_a \varphi_a$$

The potential energy of a system with continuously distributed masses is to be built in the same way, except that the sums have to be replaced by integrals. One obtains

$$\Phi = \frac{1}{2}\iint \frac{\rho\, d\tau\, \rho'\, d\tau'}{r}$$

$$\text{or} \quad \Phi = \frac{1}{2}\int \varphi\rho\, d\tau$$

This expr. is very important, for it permits the calculation of the forces that electrified bodies exert on each other.

We attach to this expression a theoretical analysis. Φ can be decomposed in such a manner that one assigns the energy $\frac{1}{2}\varphi\rho\, d\tau$ to the individual volume element. Then energy is to be assumed only where el. masses are present, e.g., on the surface. However, [p. 21] the energy can also be localized in another way. That is to say, we have

$$\Phi = \frac{1}{2}\int \varphi\rho\, d\tau = -\frac{1}{2}\int \varphi \cdot \frac{1}{4\pi}\left(\frac{\partial^2 \varphi}{\partial x^2} + \cdot + \cdot\right) = -\frac{1}{8\pi}\int \varphi \Delta \varphi\, d\tau$$

Now, $\varphi \dfrac{\partial^2 \varphi}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\varphi \dfrac{\partial \varphi}{\partial x}\right) - \left(\dfrac{\partial \varphi}{\partial x}\right)^2$

If we integrate & take into account that φ & its derivatives vanish at the limits of int., we get

$$\Phi = \frac{1}{8\pi}\int \left(\frac{\partial \varphi^2}{\partial x} + \cdot + \cdot\right) d\tau = \frac{1}{8\pi}\int (\mathfrak{E}_x^2 + \mathfrak{E}_y^2 + \mathfrak{E}_z^2)\, d\tau = \frac{1}{8\pi}\int \mathfrak{E}^2 d\tau$$

Here a volume element contributes the term $\mathfrak{E}^2 d\tau$. The energy appears localized in space. Of course, all these expressions for the total energy are equally valid. We find easily the electr. energy of an electrified conductor It is

$$\Phi = \frac{1}{2}\int \varphi\rho\, d\tau = \frac{1}{2}P\int \rho\, d\tau = \frac{1}{2}Pe = \frac{1}{2}P^2 c = \frac{1}{2}\frac{E^2}{c}$$

Application of the energy law. E in the conductor experiences an infinitely small change

1) through addition of the quantity of electricity dE, along with el. work PdE.
2) through change of shape. mech. work taken up $= dA$

The energy principle yields the equation

$$PdE + dA = d\Phi = \frac{1}{2}(PdE + EdP)$$

The mechanical work $-dA$ done by the system is

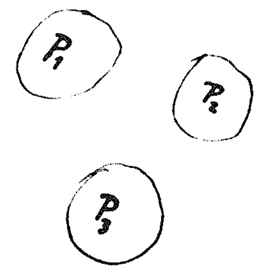
$$-dA = \frac{1}{2}(PdE - EdP)$$

If $dP = 0$, then PdE is el. work supplied. Half of it is converted into mech work. But if $dE = 0$, then

$$dA = \frac{1}{2}EdP = \frac{1}{2}Ed\left(\frac{E}{c}\right) = d\left(\frac{1}{2}\frac{E^2}{c}\right) = d\left(\frac{1}{2}EP.\right)$$

[p. 22] *Some Properties of a System of Conductors*

We imagine that the conductors 1 2 3 .. are charged How do individual potentials depend on the charges?



If φ is a solution, then $\alpha\varphi$ is also such a solution, with the surface densities, and thus also the total charges, being multiplied by α.

We start from the case

$P_1 = 1 \quad P_2 = 0$ ...... let φ_1 be this solution.

$\varphi = P_1\varphi_1$ is then also a solution.

If one defines φ_2 analogously, then

$\varphi = P_2\varphi_2$ is a solution. The φ_1 φ_2 etc. are determined by the conductors alone

$\varphi = P_1\varphi_1 + P_2\varphi_2$ .... is also a solution.

Thus, φ is homogeneous & linear in the P. The same holds for $\frac{\partial\varphi}{\partial n}$, hence also for the individual $E_1 \ldots E_n$

Thus, we get

$$\left.\begin{aligned} E_1 &= a_{11}P_1 + a_{12}P_2 + \ldots \\ E_2 &= a_{21}P_1 + a_{22}P_2 + \ldots \end{aligned}\right\} a$$

Solving for P, we get

$$\left.\begin{aligned} P_1 &= b_{11}E_1 + b_{12}E_2 + \ldots \\ P_2 &= b_{21}E_1 + b_{22}E_2 + \ldots\ldots \end{aligned}\right\} b$$

$$a_{11}P_1^2 + 2a_{12}P_1P_2 + a_{22}P_2^2$$

must not be negative[15]

$a_{11} \cdot a_{22} - 2a_{12} > 0$

sometimes another form more convenient

$$E_1 = \frac{a_{11} + a_{12}}{2}(P_1 + P_2) + \frac{a_{11} - a_{12}}{2}(P_1 - P_2)$$

$$E_2 = \frac{a_{21} + a_{22}}{2}(P_1 + P_2) + \frac{a_{21} - a_{22}}{2}(P_1 - P_2)$$

[p. 23] The coefficients satisfy a condition that we must derive. If the coefficients are constant, i.e., the position of the body remains unchanged, then ΣPdE must be a total differential. This is the case only when $b_{ik} = b_{ki}$ and $a_{ik} = a_{ki}$. This means ....

One obtains the equivalent expressions

$$\Phi = \frac{1}{2}\sum\sum a_{ik}P_iP_k$$

$$\Phi = \frac{1}{2}\sum\sum b_{ik}E_iE_k$$

From this we get

$E_1 = \frac{\partial\Phi}{\partial P_1}$ ...... where Φ is a funct of the P

$P_1 = \frac{\partial\Phi}{\partial E_1}$ ...... " " " " " " " E.

We again investigate the work performed. The latter is accompanied by a change in the coefficients. Work supplied $dA = -d\Phi$ at constant E

$$dA_m = -\frac{1}{2}(E_1dP_1 + E_2dP_2 \ldots\ldots)$$

The quantity of work is equally large if P is constant. But one has to increase the potential by dP_1 ..... In doing this el. work is supplied $dA_e = P_1dE_1 + P_2dE_2$ ......

$dA_e - dA_m = d\Phi \qquad \sum PdE - \frac{1}{2}\sum EdP = \frac{1}{2}\sum PdE + \frac{1}{2}\sum EdP$ [16]

Here $P_1dE_1 + \ldots = \frac{1}{2}(P_1dE_1 \ldots + EdP_1 \ldots\ldots)$

or $\sum E_1dP_1 = \sum P_1dE_1$

Hence the supplied el. work is[18]

$dA_e = \Sigma E_1dP_1$

$dA_e = 2dA.$

better this way:[17]

$dA_e = d\Phi + dA_m$

$dA_m = \Sigma PdE - ½\Sigma PdE - ½\Sigma EdP$

$dA_m = ½\Sigma PdE - ½\Sigma EdP$

For constant potentials

$dA_m = ½\, dA_e$

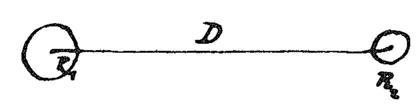
4. Example Motion of a Conductor. Plate Condenser [p. 24]

Examples.

Two spheres whose distance from each other is large compared with their radii. We calculate potentials as funct. of the quantities of electricity (approximately).

$$P_1 = \frac{1}{R_1}E_1 + \frac{1}{D}E_2 = b_{11}E_1 + b_{12}E_2$$

$$P_2 = \frac{1}{D}E_1 + \frac{1}{R_2}E_2 = b_{21}E_1 + b_{22}E_2$$



When the second sphere was moved closer, the potential of the first sphere was increased if the two were similarly charged, while the reverse took place in the opposite case.

$$\Phi = \frac{1}{2}(P_1E_1 + P_2E_2) = \frac{1}{2}(b_{11}E_1^2 + 2b_{12}E_1E_2 + b_{22}E_2^2)$$

We also calculate the constants a, which permit the evaluation of the capacities. To do this, we need only solve the above equations for E_1 & E_2. Setting $\Delta = b_{11}b_{22} - b_{12}^2$, we obtain

$$E_1 = \frac{1}{\Delta}\{b_{22}P_1 - b_{12}P_2\} = a_{11}P_1 + a_{12}P_2$$

$$E_2 = \frac{1}{\Delta}\{-b_{12}P_1 + b_{11}P_2\} = a_{12}P_1 + a_{22}P_2$$

From this we obtain the quantities of electricity for given P. If, for example, $P_2 = 0$ (the second sphere permanently grounded or connected to a casing, we have[19]

$$\frac{E_1}{P_1} = \frac{b_{22}}{\Delta} = \frac{\frac{1}{b_{11}}}{1 - \frac{b_{12}^2}{b_1b_2}} = \frac{R_1}{1 - \frac{R_1R_2}{D}}$$

The presence of the second sphere increases the capacity of the first.[20]

$$\frac{E_2}{P_1} = -\frac{b_{12}}{\Delta} \propto -\frac{b_{12}}{b_{11}b_{22}} = -\frac{R_1R_2}{D}$$

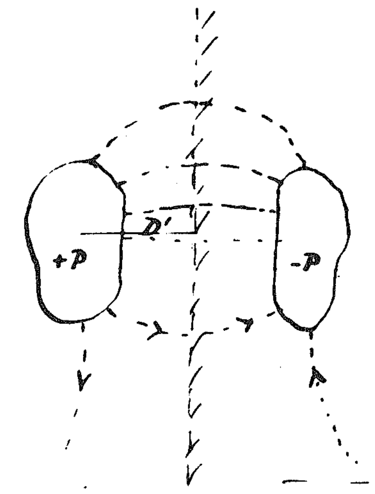
[p. 25] On the second sphere the opposite charge is produced of the approximate magnitude $P_1\frac{R_1R_2}{D}$

One can reduce the problem of the interaction between a sphere & a conducting plane to the problem of the interaction between two spheres using the principle of the electric mirror-image, which consists in the following: One sees that the case body-plane can always be reduced to the case body-symmetric body.[21] We have

$E_1 = a_{11}P - a_{12}P = (a_{11} - a_{22})P \; D = 2D'$

In our case we have, for example[22]

$$\frac{E_1}{P} = \frac{R_1}{1 - \frac{R_1R_2}{2D'}}$$



The case of two wires at a distance from each other can be treated in a manner very similar to the case of two spheres at a distance from each other, even if a conducting plane is present. All one needs for this is the potential of the electrified line. We shall not go deeper into this.

In those cases in which the field is known, another way of treating the problem, by means of $\Phi = \frac{1}{8\pi}\int \mathfrak{E}^2 d\tau$, is often more advantageous.

Plate Condenser

Only the field between the plates away from the edge is considered.[23]

$\frac{\partial \mathfrak{E}_x}{\partial x} = 0 \qquad \mathfrak{E}_x = \text{const.}$ [p. 26]

$$\Phi = \frac{1}{8\pi}\mathfrak{E}_x^2 \cdot f\delta$$

$$\Delta\varphi = -\int \frac{\partial \varphi}{\partial x}dx = \int \mathfrak{E}_x dx = \mathfrak{E}_x\delta$$

$$\text{Quantity of charge } E = \frac{\mathfrak{E}_x}{4\pi}\cdot f. \qquad 4\pi\eta = \mathfrak{E}_x$$

$$\text{Capacity} = \frac{E}{\Delta\varphi} = \frac{f}{4\pi\delta}$$

We find $\Phi = \frac{1}{2}E\Delta\varphi = \frac{1}{2}E(P_1 - P_2) = \frac{1}{2}(P_1 - P_2)^2C = \frac{1}{2}\frac{E}{c}$

We find the attractive force by constructing

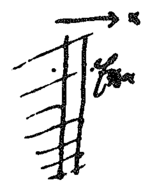
$$\left(\frac{\partial \Phi}{\partial \delta}\right)_E = \frac{1}{2}E^2\frac{\partial}{\partial \delta}\left(\frac{4\pi\delta}{f}\right) = \frac{1}{2}E^2\frac{4\pi}{f} = \frac{1}{2}E^2\cdot\frac{1}{c\delta} = \frac{1}{2}EP\cdot\frac{1}{\delta} = \frac{1}{2}P^2\frac{f}{4\pi\delta^2} = \frac{1}{8\pi}\mathfrak{E}_x^2 f$$

Thus, the force per unit surface area is $\frac{1}{8\pi}\mathfrak{E}_x$.

This last law can be derived quite generally. We seek the force that acts on the charge of unit surface. We think of the latter as being of finite thickness.

$$\text{Force} = \int \rho\mathfrak{E}_x dx \qquad \text{where } \frac{\partial \mathfrak{E}_x}{\partial x} = 4\pi\rho$$

$$= \frac{1}{4\pi}\int \mathfrak{E}_x \frac{\partial \mathfrak{E}_x}{\partial x}\, dx = \frac{1}{8\pi}\mathfrak{E}_x^2 \approx \frac{1}{2}\eta \mathfrak{E}_x$$



Thus, it is as if the outer normal force acted on the whole layer.

How does P change if one varies δ at constant E.

$P = \frac{E}{c} = E \cdot \frac{4\pi\delta}{f}$. The potential difference varies as δ. Means of increasing the potential difference by expenditure of work. How does the potential change in the case of circular plates if one increases the distance to ∞ for constant quantities of electricity?

$$E = P_1 C_1$$

$$E = P_2 C_2$$

$$\frac{P_2}{P_1} = \frac{C_1}{C_2} = \frac{\frac{f}{4\pi\delta}}{\frac{2R}{\pi}} = \frac{f}{8\delta R} = \frac{\pi}{8}\frac{R}{\delta}$$

[p. 27]

$$\Phi = \frac{1}{8\pi}\mathfrak{E}^2 f\delta = \frac{f}{8\pi}\frac{P^{2}}{\delta}$$ [24]

$$\left(\frac{P}{\delta}\right)^2$$

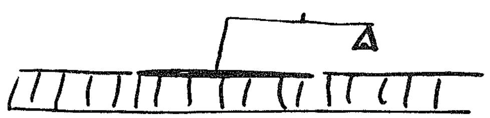
$$\frac{\partial \Phi}{\partial \delta} = -\frac{f}{8\pi}\frac{P^2}{\delta^2}$$

Application for the detection of small potential differences by electrostatic means. (Volta's experiment.)

Absolute measurement of potentials by the "guard ring" electrometer.[25]

$$\text{force} = \frac{\partial}{\partial \delta}\left(\frac{1}{2}\frac{E^2}{C}\right) = \frac{1}{2}\mathfrak{E}^2 \frac{4\pi}{f}$$

$$= 2\pi \cdot P^2 \cdot \frac{f^2}{(4\pi\delta)^2} \cdot \frac{4\pi}{f} = \frac{1}{8\pi}\frac{f}{\delta^2}P^2$$

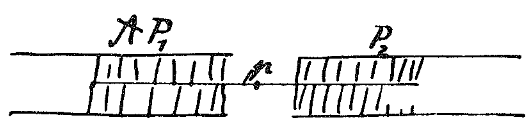


Since the force can be measured absolutely, & so too f and δ, the same is true of P.

Kelvin's Quadrant Electrometer for the Measurement of Voltages and Small Quantities of Electricity[26]

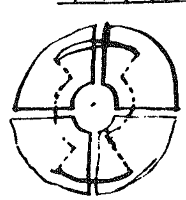
$$\Phi = \frac{1}{2}\kappa(P_1 - p)^2(a - x)$$

$$+ \frac{1}{2}\kappa(P_2 - p)^2(a + x)$$ [27]



Interests us only insofar as it is dependent on x

$$\Phi = \kappa\{(P_2 - p)^2 - (P_1 - p)^2\}x$$



$$D = \left(\frac{\partial \Phi}{\partial x}\right)_{P_p} = \kappa\{(P_2 - p)^2 - (P_1 - p)^2\}$$

Most important circuits 1) $p = P_1$ $D = \kappa(P_2 - P_1)^2$ Quadratic instrument

2) Needle at auxiliary potential p. $P_2 = P - \frac{\alpha}{2}$ $P_1 = P + \frac{\alpha}{2}$

$$D = 2\kappa\alpha(p - P)$$

If p large compared with P, then instrument is linear.

$$\frac{f}{8\pi}\frac{P^2}{\delta} \sim \frac{1}{12}\cdot\frac{\left(\frac{1}{3}\right)^2}{0.1}$$

$$\sim \frac{1}{10}$$

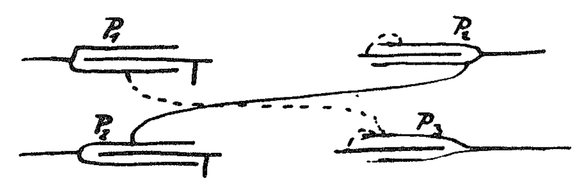
Maschinchen[28] *& Thomson's Multiplier.* [p. 28]

$$e = P_1 C$$

$$e = P C' \quad P = P_1 \frac{C}{C_1}$$

Even stronger amplification if stirrup $P = P_1 \frac{C}{c}$

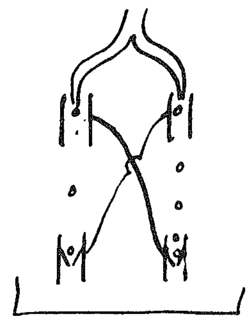
Repeat *Maschinchen*



$$P_3 = P_1\left(\frac{C}{c}\right)^2$$

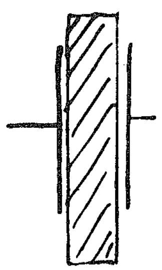
If one more connection, then Thomson's multiplier[29]

Drop multiplier



All induction machines are based on this principle.

Dielectrics



Experience. Voltage on condenser plates drops if a nonconductor is inserted between them. Conversely, if the voltage is constant, the quantity of electricity increases in a specific ratio. This ratio is characteristic of the (homogeneous) nonconductor in question. It is called the dielectric constant. <The theory that has been put forward thus far can be maintained for this case if one imagines that electricity
[p. 29] possesses limited mobility in the dielectric. Neutral molecules become dipoles>

We can now distinguish two kinds of field strength
1) Field strength between plates & dielectric or ⊥ lines of force in an arbitrary gap. ($\mathfrak{D}$)
2) Field strength in a channel connecting the plates ⊥. The latter is equal to $\frac{P_1 - P_2}{\delta} = -\frac{\partial\varphi}{\partial x}$, if x is the direction of the axis. As before, we denote this kind of field strength by $\mathfrak{E}$. The relation $\mathfrak{D} = \varepsilon\mathfrak{E}$ is generally valid everywhere in the dielectric.

It is easy to calculate the energy of such a system. We have $d\Phi = PdE$, if the plates are immovable.

But according to the special form of Gauss's law

$$\mathfrak{D} = 4\pi\sigma = 4\pi\frac{E}{f} \qquad dE = \frac{f}{4\pi}d\mathfrak{D}$$

$$P = -\int_0^\delta \mathfrak{E}dx = \mathfrak{E}\cdot\delta$$

$$d\Phi = \frac{f\delta}{4\pi}\mathfrak{E}d\mathfrak{D}$$

Integrating, we get $\Phi = \frac{V}{8\pi}\varepsilon\mathfrak{E}^2 = \frac{V}{8\pi}\mathfrak{E}\mathfrak{D}$

Accordingly, we generalize the earlier expression for the energy to

$$\Phi = \frac{1}{8\pi}\int\mathfrak{E}\mathfrak{D}d\tau = \frac{\varepsilon}{8\pi}\int\mathfrak{E}^2 d\tau$$

Intuitive representation by means of dipoles which strive to bond with one another by [p. 30]
elastic forces.

$$\left.\begin{aligned}\mathfrak{D}_x &= \varepsilon\mathfrak{E}_x\\ \mathfrak{D}_y &= \varepsilon\mathfrak{E}_y\\ \mathfrak{D}_z &= \varepsilon\mathfrak{E}_z\end{aligned}\right\}1$$

In vector notation abbreviated $\mathfrak{D} = \varepsilon\mathfrak{E}$

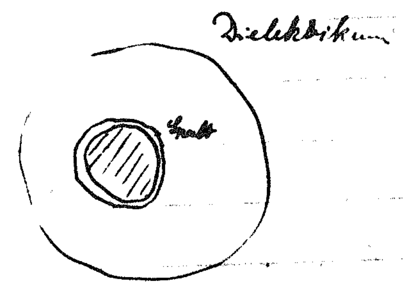
$\mathfrak{D}_x$ number of the electric lines of force (field strength) through gap perpendicular to the X axis etc. $\mathfrak{E}_x$ field strength in channel parallel to X-axis. What kinds of laws hold for the vectors $\mathfrak{D}$ and $\mathfrak{E}$ inside a dielectric?
1) $\mathfrak{E}$ derivable from the potential.

$$\left.\begin{aligned}\mathfrak{E}_x &= -\frac{\partial\varphi}{\partial x}\\ \mathfrak{E}_y &= -\frac{\partial\varphi}{\partial y}\\ \mathfrak{E}_z &= -\frac{\partial\varphi}{\partial z}\end{aligned}\right\}\ (2)$$

Intuitive model.

or, as verified by differentiation

$$\left.\begin{aligned}\frac{\partial \mathfrak{E}_z}{\partial y} - \frac{\partial \mathfrak{E}_y}{\partial z} = 0\\ \frac{\partial \mathfrak{E}_z}{\partial x} - \frac{\partial \mathfrak{E}_x}{\partial z} = 0\\ \frac{\partial \mathfrak{E}_x}{\partial y} - \frac{\partial \mathfrak{E}_y}{\partial x} = 0\end{aligned}\right\} \quad (2a)$$



2) A surface containing a great number of unbroken dipoles set up in the gap. Here Gauss's law holds.

$\int$ (normal component of the field strength in the gap) . $d\sigma = 0$

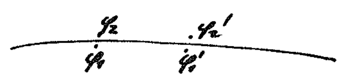
$\int \mathfrak{D}_n \, d\sigma = 0 \ldots\ldots (3)$

[p. 31] If one applies this law to a parallelepiped that is enclosed in the gap, one obtains

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 \quad \ldots . (3a)$$

What conditions hold on the boundary between two dielectrics? The constancy of φ, and hence also the constancy of the tangential components of $\mathfrak{E}$, holds here as well

$$\mathfrak{E}_{t1} = \mathfrak{E}_{t2} \ldots . \quad (2c)$$



If one chooses a relatively infinitely low cylinder whose bases are separated by the boundary surface, and applies to its boundary the generalized form of Gauss's law, one obtains

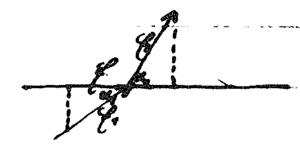
$$\mathfrak{D}_{n1} = \mathfrak{D}_{n2} \ldots\ldots (3b)$$

Refraction of the lines of force at the boundary between two media

$\mathfrak{E}_{t1} = \mathfrak{E}_{t2}$

$\varepsilon_1 \mathfrak{E}_{n1} = \varepsilon_2 \mathfrak{E}_{n2} \qquad \varepsilon_1 \operatorname{tg} \alpha_1 = \varepsilon_2 \operatorname{tg} \alpha_2$

$$\frac{\operatorname{tg} \alpha_2}{\operatorname{tg} \alpha_1} = \frac{\varepsilon_1}{\varepsilon_2}$$



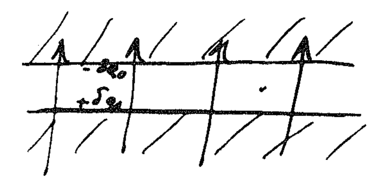
Case where movable electric quantities of spatial density ρ are also present. (1) and (2) are valid here too

3) becomes $\int \mathfrak{D}_n d\sigma = -4\pi \int \rho \, d\tau$ (3*)

$$\frac{\partial \mathfrak{D}_x}{\partial x} + \frac{\partial \mathfrak{D}_y}{\partial y} + \frac{\partial \mathfrak{D}_z}{\partial z} = 4\pi\rho \quad (3a^{x})$$

The Meaning of Dielectric Displacement [p. 32]
According to the Electron Theory[30]

The circumstance that the dielectric displacement and the electrical field strength are different in the interior of insulators has been attributed to the limited motion of elastically bound electricity. We investigate the meaning of $\mathfrak{D}$ according to this conception. Positive as well as negative el. in nonelectrified state density ρ_0. $\mathfrak{D} - \mathfrak{E}$ is produced by the field in the gap through the action of bound electricity. If δ is now the displacement of the positive el. in the insulator, then



$\delta \rho_0$ is posit. coating below
$-\delta \rho_0$ neg. coating on top.

Each sends out $4\pi\delta\rho_0$ lines of force, hence, $2\pi\delta\rho_0$ to one side Both together $4\pi\delta\rho_0$ in the gap. Thus, we have

$$\mathfrak{D} - \mathfrak{E} = 4\pi\delta\rho_0$$

Nothing changes here if we assume that the electricity in the dielectric is distributed in discrete quantities $\pm e$. Then $\rho_0 = ne$

$$\delta\rho_0 = n\underbrace{e\delta}_{\mu} = n\mu = \mathfrak{P}$$

Thus one obtains

$$\mathfrak{D} - \mathfrak{E} = 4\pi\mathfrak{P}.$$

[p. 33]

Electrostatic Energy

$\frac{1}{8\pi}\int(\mathfrak{E}_x\mathfrak{D}_x + \mathfrak{E}_y\mathfrak{D}_y + \cdot)d\tau = \Phi = \frac{1}{2}\int\varphi\rho d\tau$ | Extended over ∞ space.

Another form.

$$-\int\left(\frac{\partial\varphi}{\partial x}\mathfrak{D}_x + \cdot + \cdot\right) = +\int\left(\varphi\frac{\partial\mathfrak{D}_x}{\partial x} + \cdot + \cdot\right) = 4\pi\int\varphi\rho\, d\tau.$$

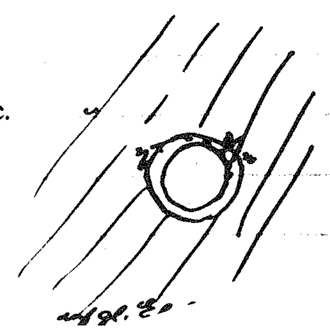
Thus, the second form of the energy expression also holds unchanged. The uniqueness proof for the conductor problem in the case where arbitrary uncharged dielectrics are present is also easy to carry out. We think of the dielectrics as being *uniformly* distributed. In that case φ & $\frac{\partial\varphi}{\partial x}$ etc. are constant in the entire space except on the conductor surfaces.

$$\int(\mathfrak{E}_x\mathfrak{D}_x + \cdot + \cdot) = +\int(\varphi\mathfrak{D}_n)d\sigma + \int\varphi\rho d\tau$$

In the domain of integration $\rho = 0$. on the boundaries $\varphi = 0$ for difference solution. Thus, the left side = 0, which is a sum of positive magn.

Charged sphere. Generalization of Coul. law. Forces calculable from Φ using the energy principle.

$$K = \frac{ee'}{r^2}\cdot\frac{1}{\varepsilon}$$



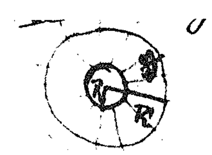
<Energy of a> charged sphere <*P*> in the dielectric.

$4\pi\sigma = \mathfrak{E}_{PL} = \mathfrak{D}_{\text{dielectr.}} = \mathfrak{E}_D\varepsilon$

$\mathfrak{E}_D = \mathfrak{E}_P\cdot\frac{1}{\varepsilon}$ $\quad\varphi_D = \varphi_L\cdot\frac{1}{\varepsilon}$

$\Phi = \frac{1}{\varepsilon}\Phi_L$

It requires less energy to charge a sphere in the dielectric to the same quantity of electricity, more energy to charge it to the same voltage.

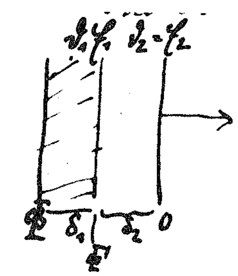


$\mathfrak{D}_1 = \mathfrak{D}_2 \quad 4\pi R^2\mathfrak{D} = 4\pi e$

$\mathfrak{D} = \frac{e}{r^2} \quad \mathfrak{E}_{(1)} = \frac{1}{\varepsilon}\frac{e}{r^2} \quad \mathfrak{E}_{(2)} = \frac{e}{r^2} = -\frac{\partial\varphi}{\partial r}$

$$\Phi = \int_\infty^{R'}\frac{e}{r^2}dr + \int_{R'}^{R}\frac{1}{\varepsilon}\frac{e}{r^2}dr = e\left\{\frac{1}{R'} + \varepsilon\left(\frac{1}{R} - \frac{1}{R'}\right)\right\}$$

Plate condenser partly with air, partly with dielectric [p. 34]



$\mathfrak{E}_1 = -\frac{\partial\varphi}{\partial x} \quad \mathfrak{E}_2 = \frac{\partial\varphi}{\partial x}$

$\mathfrak{D}_1 = \varepsilon\mathfrak{E}_1$

<$4\pi\sigma = \mathfrak{D}_1$> boundary condition[31] $\varphi = \Phi$ for $x = 0$

$\varphi = 0$ for $x = \delta_1 + \delta_2$

$\mathfrak{D}_1 = \mathfrak{E}_2$ for $x = \delta$

Vectors spatially constant

$$\Phi - \Phi' = -\int_0^{\delta_1}\frac{\partial\varphi}{\partial x}dx = \mathfrak{E}_1\delta_1$$

$$\Phi' - 0 = -\int\frac{\partial\varphi}{\partial x}dx = \mathfrak{E}_2\delta_2$$

$$\Phi = \mathfrak{E}_1\delta_1 + \mathfrak{E}_2\delta_2$$

$$\mathfrak{D}_1 = \mathfrak{E}_2 = \varepsilon\mathfrak{E}_1$$

$$\Phi = \mathfrak{E}_2\left(\delta_2 + \frac{\delta_1}{\varepsilon}\right) = \mathfrak{D}_1\left(\delta_2 + \frac{\delta_1}{\varepsilon}\right)$$

Charges $-\frac{1}{4\pi}\mathfrak{E}_2 \quad \frac{1}{4\pi}\mathfrak{D}_1 = \eta$, thus, equally large.

$$\Phi = 4\pi\eta\left(\delta_2 + \frac{\delta_1}{\varepsilon}\right) = 4\pi\frac{E}{f}\left(\delta_2 + \frac{\delta_1}{\varepsilon}\right)$$

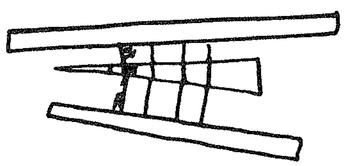
$$C = \frac{f}{4\pi\left(\delta_2 + \frac{\delta_1}{\varepsilon}\right)}$$

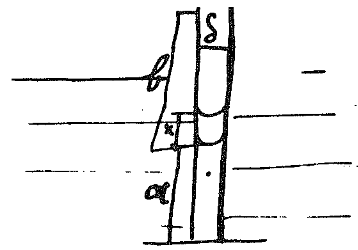
static methods for the determination of the dielectric const. Comparison of condenser potentials when the charge is the same.

Force proportional to ε when voltage given. From this, the dielectric constant of liquids.

[p. 35] Rising of liquids between plates. Perot, refraction of the lines of force.[32]

$$\frac{tg\ \alpha}{tg\ \beta} = \frac{\varepsilon}{\varepsilon_0}$$





α = height of rise without field.
$\alpha + x$ = height of rise with field

Pot. energy of gravity $\delta \cdot (<\alpha+>x) \cdot \rho \cdot \frac{\alpha + x}{2g}$

$$\Phi_g = \delta \frac{(<\alpha> + x)^2}{2} \rho g$$

$$\Phi_e = \frac{1}{8\pi} \mathfrak{E}^2 (\delta(\alpha + x)\varepsilon + \delta(b - x)\varepsilon_0)$$

$d\Phi_e$ = work of the el. forces
$-d\Phi_e$ = work of the grav. forces
Sum must be zero.

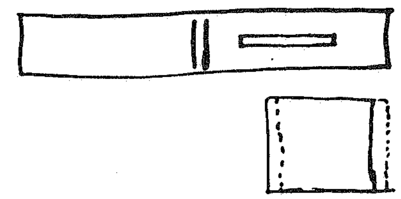
$$\delta(<\alpha+>x)\rho g - \frac{1}{8\pi}\mathfrak{E}^2 <\delta> (\varepsilon - \varepsilon) = 0$$

absolute measurement of $(\varepsilon - 1)$.
Better directly with force.

Volta Effect. Electric Double Layer.
Magnetism.

Coul.'s Law Unit of Pole Strength. Potential Laplace's Theorem

Intuitive meaning of the magnetization constants. Let us have a homogeneous isotropic material in the shape of a bar. Let it be uniformly magnetized. Displacement δ



Density of the polarization electricity[33] of either sign ρ_0 How large are H and $\mathfrak{B}$? Material surface perpendicular to the x axis. $\rho_0\delta$ positive <electric-ity> magnetism has traversed unit surface area Coatings of density $\rho_0\delta$. When surface slanted, then $\rho_0\delta\cos\varphi$ exit per unit surface. If molecular model, then $\rho_0 = \mu \cdot n$ Density of coating $\mu n \delta\cos\varphi = \mathfrak{Q}\cos\varphi$ $\mathfrak{Q}$ polarization. Channel walls do not have magnetic covering. End surfaces can [p. 36] be neglected. Thus, magnetic field str. in the interior of the channel the same as in the channel.

But in the gap the coverings do send out lines of force. From Gauss's law directly $B - \mathfrak{H} = 4\pi <\mathfrak{Q}>$. Exactly as with dielectrics. There are no true magnetic masses. From this it follows that

$$\int \mathfrak{B}_n d\sigma = 0. \quad or \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

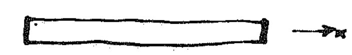
$\mathfrak{H}$ derivable from a potential. Potential of the densities of bound magnetism.

$$\mathfrak{H} = -\frac{\partial \varphi}{\partial x} \text{ etc.}$$

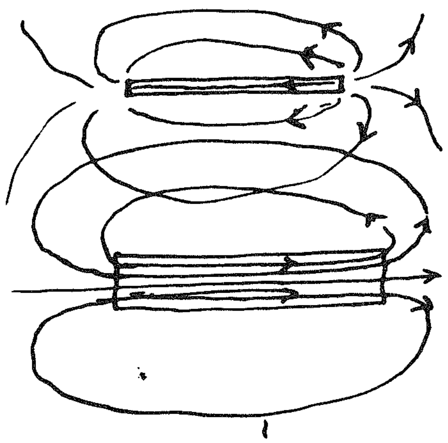
These are our fundamental laws.
Where is the density of the bound magnetism located? Surface in subst.

$$4\pi \int \rho_g d\tau = \int \mathfrak{H}_n d\sigma = -\int \mathfrak{P}_n d\sigma$$

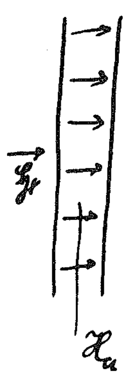
$$\rho_g = -\left(\frac{\partial \mathfrak{P}_x}{\partial x} + \frac{\partial \mathfrak{P}_y}{\partial y} + \frac{\partial \mathfrak{P}_z}{\partial z}\right)$$

Parallel magnetized iron bar 

Magnetic coverings of bound magnetism $= f\mathfrak{P}_x$
The fields $\mathfrak{H}$ and $\mathfrak{B}$ are here independent of each other in the magnet $\mathfrak{H}$ and $\mathfrak{B}$ are differently oriented in the magnet.



[p. 37] Magnetic force exerted on each other by very closely neighboring surfaces.



$$\eta_g = \frac{1}{4\pi}(\mathfrak{H}_a - \mathfrak{H}_i)$$

$$4\pi\rho = \frac{\partial \mathfrak{H}_x}{\partial x} + \underset{0}{\frac{\partial \mathfrak{H}_y}{\partial y}} + \underset{0}{\frac{\partial \mathfrak{H}_z}{\partial z}}$$

$$\int \rho \mathfrak{H}_x dx = \frac{1}{4\pi}\int \mathfrak{H}_x \frac{d\mathfrak{H}_x}{dx} dx = \frac{1}{8\pi}(\mathfrak{H}_a^2 - \mathfrak{H}_i^2)$$

$$K = \frac{1}{8\pi}(\mathfrak{B}_i^2 - \mathfrak{H}_i^2)$$

If iron, then $\mathfrak{B}_i$ larger than $\mathfrak{H}_i$, so that approx. $K = \frac{1}{8\pi}\mathfrak{B}_i^2$.

This must be eq. to the magnetic energy of the unit volume.
Can be very large. $\mathfrak{B}_i = 20{,}000$ $K \sim 2.10^2$ = ca 20 kg per cm^2.

Energy of the Magnetic Field

a) in vacuum $\sum\sum \frac{\mu\mu'}{r} = \frac{1}{2}\sum \mu\varphi$

This is $g\frac{1}{8\pi}\mathfrak{H}^2$

b) If $\mu \neq 1$, then this energy has another value. One can calculate this value by taking into account that work must also be expended for the displacement.

The posit. magn. masses of the unit volume are subjected to the total force $\mathfrak{H}\rho_0$ The work expended on the displ. $d\delta$ is $\mathfrak{H}\rho_0 d\delta$. This is equal to $\mathfrak{H}d\mathfrak{P}$ because one can set $\rho_0\delta = \mathfrak{P}$.

In the process, magnetic energy in the vacuum increases by $\frac{1}{4\pi}\mathfrak{H}d\mathfrak{H}$.

The two combined

$$\frac{1}{4\pi}\mathfrak{H}d(\mathfrak{H} + 4\pi\mathfrak{P}) = \frac{1}{4\pi}\mathfrak{H}d\mathfrak{B}.$$

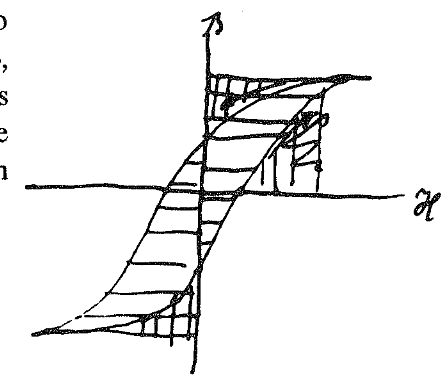
If $\mathfrak{B} = \mu\mathfrak{H}$, then this is integrable $\frac{1}{4\pi}\int \mu\mathfrak{H}d\mathfrak{H}$

If μ = const, then

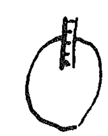
$$\frac{1}{8\pi}\mu H^2$$ [p. 38]

per unit volume. As a matter of fact, one may designate this energy as "magnetic."

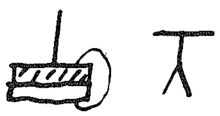
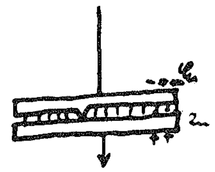
But things are different in the case where no relat. exists between $\mathfrak{H}$ and $\mathfrak{B}$. In that case, too, $Hd\mathfrak{B}$ is the work supplied to unit volume. But this work need not represent an available store. Surface of the "hysteresis curve" represents the energy lost in a cyclic process. This energy is converted to heat.



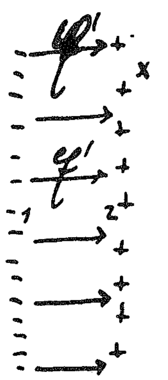
The Volta Effect - Electromotive Forces



In the arrangement shown in the accompanying sketch, one observes an el. field between the plates. Such a field would not to be expected according to the theory employed so far. Potential diff. cannot arise in the interior of metals. Hence they must arise on the boundary surfaces. Let us first assume that the potential jump occurs more or less at the contact surface of the metals—later on this will turn out not to be valid. Volta discovered.

 Mechanics of the effect 

Contradiction with the theory employed so far. How can the latter be extended so
[p. 39] that in agreement with experience At the surface of separation, electricities are acted upon by a force that separates them. We will conceive of this as a field that has an external source, not originating from el. masses.



This "impressed" force seeks to move positive electr. to the right.[34] Equilib. can exist only if the effect of $\mathfrak{E}'$ is compensated by an opposite electrost. field.

$\mathfrak{E} + \mathfrak{E} = 0 \quad \mathfrak{E}' - \frac{\partial\varphi}{\partial x} = 0 \quad \varphi_2 - \varphi_1 = \int \mathfrak{E}'\,dx.$

Thus, there is a potential jump at the surface. How is it produced?

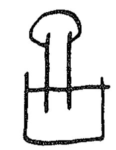
$$\frac{\partial \mathfrak{E}_x}{\partial x} = 4\pi\rho = -\frac{\partial \mathfrak{E}'}{\partial x}$$

Thus, we have two opposite coverings. If $\mathfrak{E}'$ is constant inside the layer, then these coverings are planar $(\eta)\,|\mathfrak{E}| = 4\pi\eta$ Gauss's law.

$\int \mathfrak{E}'dx = \Delta\varphi = \mathfrak{E}'\delta = 4\pi\underbrace{\eta\delta}_{\zeta}\ \zeta$ moment of the unit surface area of the double layer

Double layer <corresponds> not an arbitr. theory but demanded directly by experience. Is very dependent on the constitution of the surface—especially water layer. Can be removed almost completely with removal of the latter. Thus, is located in the surface facing the air.

[p. 40] If instead of air, water between the plates, then also field.



But because water conductor, el. moves in water. Current arises. In accordance with the law of conservation of el., such a current must also flow in the metals, so that no excess would arise anywhere. Chem. processes on electr. Since we have already meas. the el. unit, the unit of the el. current is also def. (Number of electrost. units flowing through the conductor per second. The direction of the current is the direction in which the positive electricity flows.

Magnetic Field of Currents

Current acts on magnetic needle. What is the constitution of the magnetic field outside the conductor?

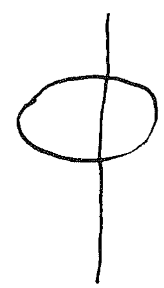
Let the conductors be surrounded by vacuum (or air). For such a case we have found that

$$\mathfrak{H}_x = -\frac{\partial\varphi}{\partial x}\,\cdot\cdot$$

$$\frac{\partial \mathfrak{H}_x}{\partial x} + \frac{\partial \mathfrak{H}_y}{\partial y} + \frac{\partial \mathfrak{H}_z}{\partial z} = 0$$

If the concept of magnetic field has a general meaning, then these equations must hold here as well.

$\mathfrak{H}$ derivable from a potential. In such a case we have seen until now that the line integral of the (magn.) field strength over a closed curve always vanished.



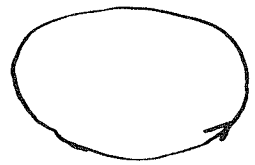
But it does not take much to see that the magn. lines of force surround [p. 41]
an el. current. Thus, if we form the line integral $\int(\mathfrak{H}_x dx + \mathfrak{H}_y dy + \mathfrak{H}_z dz) = \int \mathfrak{H}ds \cos(\mathfrak{H}ds)$, we certainly do not obtain zero.

This notwithstanding, our above formulas may be right

$$\int(\mathfrak{H}_x dx + \cdot\cdot+\cdot) = -\int d\varphi = \varphi_1 - \varphi_2$$

This quantity must vanish, then, for a closed path only if φ is a *single-valued* spatial function. How must fields be constituted for φ to become multivalued? In order to resolve this, we investigate the closed line integral of an arbitrary vector.

Stokes's Theorem

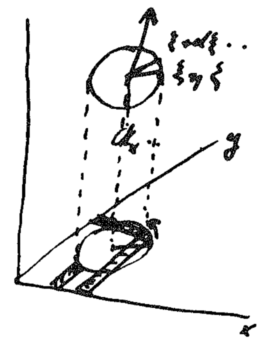


Vector $\mathfrak{A}_x \mathfrak{A}_y \mathfrak{A}_z$

Line integral $\int \mathfrak{A}_x dx + \mathfrak{A}_y dy + \mathfrak{A}_z dz$

Decomposed into so many such integrals over ∞ small surfaces which can be regarded as planar.

Over this $\int \mathfrak{A}_x dx$



$$\int \mathfrak{A}_x dx = \int \left(\mathfrak{A}_{x_0} + \frac{\partial \mathfrak{A}_x}{\partial \xi_0} \xi + \frac{\partial \mathfrak{A}_x}{\partial \eta_0} \eta + \frac{\partial \mathfrak{A}_x}{\partial \zeta_0} \zeta \right) d\xi$$

$$= \mathfrak{A}_{x_0} \underbrace{\int d\xi}_{0} + \frac{\partial \mathfrak{A}_x}{\partial \xi} \underbrace{\int \xi \, d\xi}_{0} + \frac{\partial \mathfrak{A}_x}{\partial \eta} \underbrace{\int \eta \, d\xi}_{-d\sigma \cos nz} + \underbrace{\frac{\partial \mathfrak{A}_x}{\partial \zeta} \zeta \, d\xi}_{+d\sigma \cos ny}$$

$\int_{d\sigma} \mathfrak{A} \, dx = +d\sigma$	$\frac{\partial \mathfrak{A}_x}{\partial \zeta} \cos ny - \frac{\partial \mathfrak{A}_x}{\partial \eta} \cos nz$	$\left(\frac{\partial \mathfrak{A}_z}{\partial y} - \frac{\partial \mathfrak{A}_y}{\partial z} \right) \cos nx$
————————	$\frac{\partial \mathfrak{A}_y}{\partial \xi} \cos nz - \frac{\partial \mathfrak{A}_y}{\partial \zeta} \cos nx$	————————————
————————	$\frac{\partial \mathfrak{A}_z}{\partial \eta} \cos nx - \frac{\partial \mathfrak{A}_z}{\partial \xi} \cos ny$	————————————

The integral is thereby converted into a surface integral.

[p. 42] *Elementary Derivation of the Properties of the Magnetic Field*

For a field of permanent magnets we have

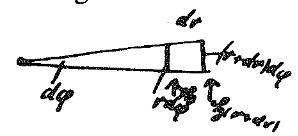
$$\frac{\partial \mathfrak{H}_x}{\partial x} + \frac{\partial \mathfrak{H}_y}{\partial y} + \frac{\partial \mathfrak{H}_z}{\partial z} = 0$$

and

$$\mathfrak{H}_x = -\frac{\partial \varphi}{\partial x} \quad \mathfrak{H}_y = -\frac{\partial \varphi}{\partial y} \quad \mathfrak{H}_z = -\frac{\partial \varphi}{\partial z}$$

If φ is a single-valued function, this means that the line integral of $\mathfrak{H}$ along a closed curve is zero.

We now assume that the lines of force around a rectilinear current path are circles. How must then the field decrease with the distance? In the space that is simply connected & outside the circuit, the field of magn. & the field of current shall not be distinguishable from each other. How must then the field strength depend on the dist?



$$\text{Line integral} = \mathfrak{H}(\overset{r'}{\overbrace{r + dr}}) \cdot (\overset{r'}{\overbrace{r + dr}}) \, d\varphi - \mathfrak{H}(r) r \, d\varphi = 0$$

$$\mathfrak{H}(r') \cdot r' = \mathfrak{H}(r) \cdot r$$

If we let r vary for constant r', then

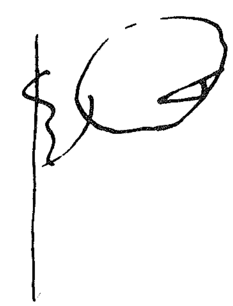
we obtain $\mathfrak{H}(r) \cdot r = \text{const.}$ $\mathfrak{H}(r) = \dfrac{\text{const.}}{\text{r}}$

This law is confirmed by experience. The const. depends on the strength of the current. It can serve as a measure of the current strength. We stipulate const $= 2i$, and thereby obtain a definition for the current strength

$$\mathfrak{H} = \frac{2i}{r}$$

i is then equal to 1 if the current produces field strength 2 at dist. of 1 cm. This dependence on r is confirmed by experience.

If we integrate $\int \mathfrak{H} ds$ along a circle around the current path, we obtain $\int \frac{2i}{r} \cdot r d\varphi = 4\pi i$, [p. 43] thus independent of r. But this is valid not only for a circular path but for any arbitrary path.



$$\mathfrak{H}_s ds = \mathfrak{H}_t \cdot r d\varphi = \frac{2i}{r} r d\varphi = 2i d\varphi$$

$$\int \mathfrak{H}_s ds = 4\pi i.$$

Potential[35] $\int \mathfrak{H}_s ds = -\int \frac{\partial \varphi}{\partial s} ds = 2i \int d\vartheta$

$$d\varphi = -2i d\vartheta$$

$$\varphi = -2i\vartheta + \text{const.}$$

Now, φ is not a single-valued function, because ∞ many angles ϑ belong to one location. If many currents, then pot ∞ multivalued Differential equations of the magnetic force derived from that.

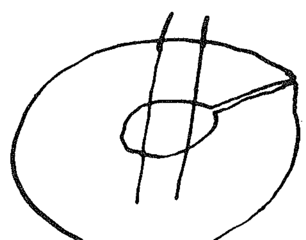
Arbitrarily Distributed Currents (Rigorous Analysis)

We start off from Stokes's theorem

$$\int \mathfrak{H}_x\,dx + \mathfrak{H}_y\,dy + \mathfrak{H}_z\,dz = \int\left\{\left(\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z}\right)\cos nx + \cdot + \cdot\right\} d\sigma$$

If in all points of the plane $\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z}$ etc $= 0$,

then the integral vanishes over every closed curve. But this is not at all the case if the current is twisted around. In that case, however, the integral over the curve shown in the sketch vanishes. $\int \mathfrak{H} ds$ cos $\mathfrak{H}ds$ is independent of the integration path. In engineering this quantity is called the "magneto-motive force". We set this quantity equal to $4\pi i$. We set the current density to be $i_x i_y i_z$, then the electricity flowing through $d\sigma$ per unit time is ($i_x \cos nx$ + $i_y \cos ny$ + $i_z \cos nz$) $d\sigma$, so that we have



$$4\pi \int (i_x \cos nx + \cdot + \cdot)\,d\sigma = \int\left\{\left(\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z}\right\} \cos nx + \cdot + \cdot\right\} d\sigma$$

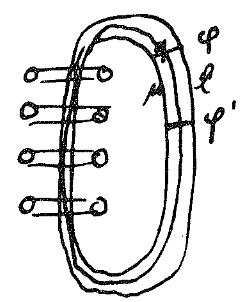
Holds also for an infinitely small surface.

$$4\pi i_x = \frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z}$$

- - - - - - - - - - - - - - -
- - - - - - - - - - - - - - -

[p. 44] *Applications of the Integral Law*

$4\pi i = \int \mathfrak{H}_s\, ds$, if current wound once
$4\pi n i = \int \mathfrak{H}_s\, ds$ " " " n times



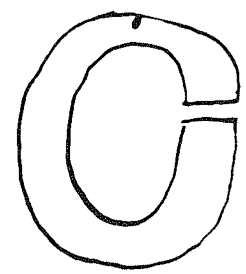
$$\mathfrak{H}_s = \frac{\varphi - \varphi'}{l}$$

$$\Sigma\,(\varphi - \varphi') = 4\pi n i$$

$$= \Sigma\, \mathfrak{H}_s l = \Sigma \frac{1}{\mu} \mathfrak{B}_s l$$

$$= \Sigma \frac{F}{f} \cdot \frac{l}{\mu} = F \cdot \Sigma \frac{l}{\mu f} = 4\pi n i$$

Σ is called the magnetic resistance of the line-of-force tube. $F \int \mathfrak{B}_n df$ = flux



Solenoid inside & outside. Pole str. of the solenoid

$4\pi n i = \mathfrak{H} \cdot l$ $\frac{4\pi n i}{l} q$ = number of lines of force.

$\frac{n i q}{l}$ = pole strength.

Determination of the field when the position of the currents is given. $\mathfrak{H}_1$ & $\mathfrak{H}_2$ two solutions. Difference $\mathfrak{H}$.

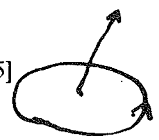
Everywhere $\frac{\partial \mathfrak{H}_z}{\partial y} = \frac{\partial \mathfrak{H}_y}{\partial z}$ ...

Then $\int (\mathfrak{H}_x dx + \cdot + \cdot)$ indep. of integr path $= -\varphi$. Then $\mathfrak{H}_x = -\frac{\partial \varphi}{\partial x}$ ... Thus, in the entire space φ dependent on single-valued potential.[36] $\int \left(\frac{\partial \varphi^2}{\partial x} + \cdot + \cdot\right) d\tau = -\int \varphi \Delta \varphi\, d\tau = 0$

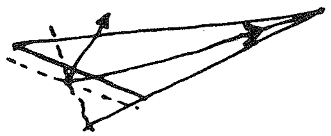
(at least if no iron pres) Thus, φ = const. Thus, uniquely determined holds also if bodies with $\mu \neq 1$ are present.

Magnetic Potential of a Current (Ampère)

What is sought is the potential function, which changes by $4\pi i$ for one circle around the current.

[p. 45] 

Double layer considered[37]



$$\int d\sigma \cdot \eta \cdot \frac{\partial \frac{1}{r}}{\partial n} \delta = \zeta \int d\sigma \frac{1}{r^2} \cos nr$$
$$= \zeta d\kappa$$

For finite angle $\zeta\kappa$
Potential changes by $4\pi\xi$ for one revolution, no matter at which point one starts and to which point one gets. Current replaceable by double layer of moment ζ. Holds only outside the double layer.

Action-at-Distance of Circuits from Maxwell's Equations without Iron

$$4\pi \mathfrak{i}_x = \frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z} = -\Delta\Gamma_x \qquad \mathfrak{H}_x = \frac{\partial \Gamma_z}{\partial y} - \frac{\partial \Gamma_y}{\partial z}$$

$$4\pi \mathfrak{i}_y = \frac{\partial \mathfrak{H}_x}{\partial z} - \frac{\partial \mathfrak{H}_z}{\partial x} \qquad \frac{\partial}{\partial z}\mathfrak{H}_y = \frac{\partial \Gamma_x}{\partial z} - \frac{\partial \Gamma_z}{\partial x}$$

$$4\pi \mathfrak{i}_z = \text{---------} \qquad \frac{\partial}{\partial y}\mathfrak{H}_z = \frac{\partial \Gamma_y}{\partial x} - \frac{\partial \Gamma_x}{\partial y}$$

$$\Delta\Gamma_x = -4\pi \mathfrak{i}_x$$

Rel. same as betw. pot. & el. density. Thus,

$$\Gamma_x = \int \frac{\mathfrak{i}_x d\tau}{r}$$

$$\Gamma_y = \int \frac{\mathfrak{i}_y d\tau}{r} \qquad \text{In fact } \frac{\partial \Gamma_x}{\partial x} + \cdot + \cdot = 0$$

$$\text{as consequence of } \frac{\partial \mathfrak{i}_x}{\partial x} + \cdot + \cdot = 0$$

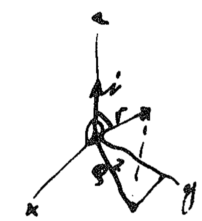
Represented as the sum of the distant-actions of elements.

 $qds = d\tau$

$$\mathfrak{H}_x d\tau = d\tau \frac{\mathfrak{i}_y z - \mathfrak{i}_z y}{r^3} \qquad \mathfrak{H}_x ds = ds \frac{\mathfrak{i}_y z - \mathfrak{i}_z y}{r^3} = \mathfrak{i}\, ds \frac{\beta z - \gamma y}{r}$$

$$\mathfrak{H}_y d\tau = d\tau \text{-------} \qquad \text{---------------------------}$$

---------------- ---------------------------

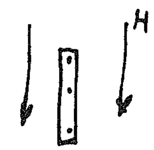


Is ⊥ to $\mathfrak{i}$ & r Choose $\mathfrak{i}$ and r as in Fig.

Then $\mathfrak{H} = \frac{d\tau}{r^3}\mathfrak{i} \cdot \rho = \frac{d\tau\, \mathfrak{i}}{r^2} \cdot \sin(\mathfrak{i}r)$ Interpretation of the vector product.

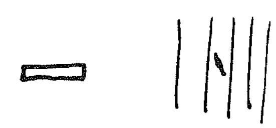
Media with permeability present. $\mathfrak{B}_x = \frac{\partial \Gamma_z}{\partial y} - \frac{\partial \Gamma_y}{\partial z} - \frac{\partial \varphi}{\partial x}$ [p. 46]

Galvanometer with Earth Field Intensity of the Latter
Magnetometer for the Determination of μ[38]



$$-MH \sin x = I\frac{d^2x}{dt^2} \qquad x = A \sin 2\pi\frac{t}{T}$$

For small oscillations $\frac{d^2x}{dt^2} + \frac{MH}{I}x = 0$



$\left(\frac{2\pi}{T}\right)^2 = \frac{MH}{I}$ For meas [---] I & $I + I'$

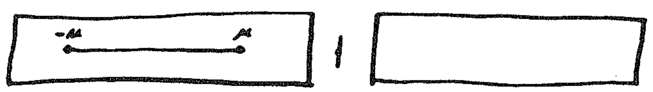
$\frac{M}{H}$ can be determined

From this M & H separately (Gauss).
If H is known, then current strength with tangent galvanometer.[39]

$$\frac{2\pi R}{R^2} i = H_i$$

From this H_i & thus also i.

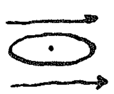
Magnetometer



If infinitely thin magnetizable bar inside, then deflection owing to transverse field.

$\frac{4\pi i n}{l} = \mathfrak{H}_s$ known.

$$\mathfrak{B} = \mu\mathfrak{H}_s \qquad \mathfrak{J} = (\mu - l)\mathfrak{H}_s$$



$$\mathfrak{m} = \frac{(\mu - 1)}{4\pi}\mathfrak{H}_s \cdot f$$

$$\mathfrak{m}\left(\frac{1}{r^2} - \frac{1}{r'^2}\right) = \mathfrak{H}_a$$

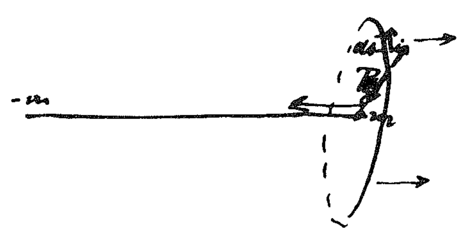
If the little bar is of finite thickness, then demagnetization factor

$$H_r = H - F\mathfrak{J} = H - F\frac{\mu - 1}{4\pi}H_r \qquad H = H_r\left(1 + \frac{\mu - 1}{4\pi}F\right)$$

$\mathfrak{J} = \frac{\mu - 1}{4\pi}H_r$ $\quad M = \mathfrak{J}\cdot V$ connection more indirect.

[p. 47] $\mathfrak{J} = \frac{\kappa}{1 + \kappa F}.H \qquad I = \mathfrak{J}\cdot \text{Vol.} \qquad F =$

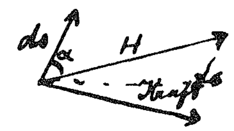
Ponderomotive Force on Element of Current



System cannot start moving by itself. Action & reaction are equal to one another.

$\frac{i.2\pi R}{R^2}\cdot m$ = force on magnetic pole, and thus, conversely, force on current. Thus, force on element of current $= i\frac{m}{R^2}ds = \underline{iHds}$.

No force in the direction of the element. General formulation. Force ⊥ to i ⊥ to H. If no right angle between H & ds, then only the component of H perpendicular to ds effective.



$H\cdot ids\cdot \sin\alpha$

We have to form the so-called vect. product of ds and H

$dK_x = i(dy\,\mathfrak{H}_z - dz\,\mathfrak{H}_y) \qquad dK_x = dt(i_y\mathfrak{H}_z - i_z\mathfrak{H}_y)$

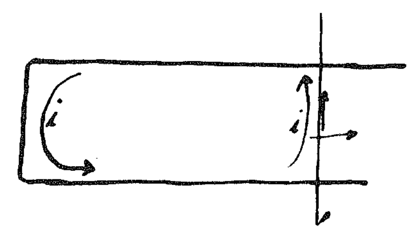
- - - - - - - - - - - - - - - - - - - ....

- - - - - - - - - - - - - - - - - - - ....

If instead of air or vacuum subst with permeability μ, then ponderomotive force dep. on $\mathfrak{B}$. Again *force* $= \frac{i2\pi R}{R^2}m$, but $\frac{\mu\pi m}{R^2} = \mathfrak{B}$.

Deprez-D'Arsonval instruments.[40]

Total force on finite conductor by integration.[41] [p. 48]



$l\cdot i\cdot\mathfrak{H}\cdot\delta = i\cdot\Delta N.$

Work of the ponderomotive forces = increase in the number of lines of force · current strength. Flexible circuit seeks maximum extension. In general, work on element of current = number of the lines of force intersected. Force vector[42]

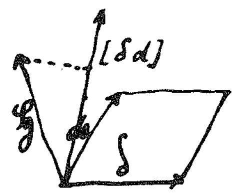
$$\begin{array}{l|l} i(dy\,\mathfrak{H}_z z - dz\,\mathfrak{H}_y)\cdot & \delta_x \\ i(dz\,\mathfrak{H}_x - dx\,\mathfrak{H}_z) & \delta_y \\ i(dx\,\mathfrak{H}_y - dy\,\mathfrak{H}_x) & \delta_z \end{array}$$

Multiplied by components of displacement $\delta_x\delta_y\delta_z$ yields work.
This can also be arranged in the following way

$i(\delta_y dz - \delta_z dy)$

$i(\text{----------})$

$\mathfrak{H}_x$

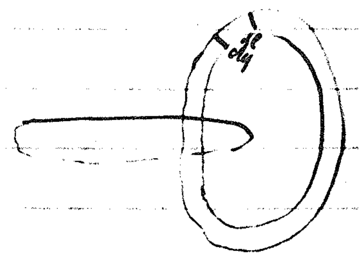


This proves the theorem.

A circuit with given i seeks to orient and deform itself in such a way that the number of
[p. 49] lines of force it cuts becomes a maximum. Thus, the forces acting on circuits have <potential> funct. that plays role of pot. en equal to iN, where N thus directed is positive, like the field generated by the current.

Magnetic energy of a circuit.

$$\frac{1}{8\pi}\sum \mathfrak{B}\mathfrak{H}\, q dl = \frac{N}{8\pi}\int \mathfrak{H}\, dl = \frac{Ni}{2}$$



Electrostatic & Electromagn. Measure of the Current Strength & Quantity of El.

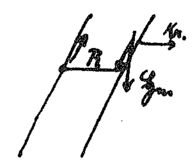
In electrostatics we derived an absolute measure for the quantity of el. & potential difference/electrostatic measure. $\mathfrak{E} = 4\pi\sigma$

 Force $= \mathfrak{E}\cdot\frac{\sigma}{2}\cdot f = \frac{1}{8\pi}\mathfrak{E}^2 f = 2\pi\sigma^2 f = 2\pi\frac{E^2}{f}$

$$E_s = \frac{1}{\sqrt{2\pi}}\sqrt{\text{force}\cdot f} \approx M^{\frac{1}{2}} l^{\frac{1}{2}+\frac{3}{2}} t^{-1} \quad i_s = M^{\frac{1}{2}} l^{\frac{1}{2}+\frac{3}{2}} t^{-2}$$

The quantity of electricity can also be measured electrodynamically as $\int i_m dt = E_m$

Dimension of the magnetically measured current: [Force]



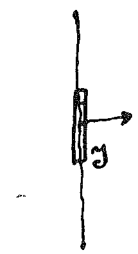
$$\frac{2i_m}{R} = H \qquad H_m i_m l = \text{Force} = \frac{2i_m^2 l}{R} \qquad M\frac{L}{T^2} = |i_m^2|$$

$$\{i_m\} = M^{1/2}L^{1/2}T^{-1}$$

$$\left\{\frac{i_s}{i_m}\right\} = \frac{L}{T}$$

Deprez-D'Arsonval.[43]  $\mathfrak{H} li \cdot 2nR = D = \Theta x$ equilibrium. [p. 50]

abs. measurement of quantities of electricity



$$\mathfrak{H} = \frac{2\pi i}{R} \quad \mathfrak{H}M = I\frac{d^2x}{dt^2} \quad \int_t^0 \mathfrak{H}dt = \frac{I}{M}\left|\frac{dx}{dt}\right|_T = \frac{2\pi}{R}\int idt$$

$$\int idt = \frac{R}{2\pi}\frac{I}{M}\left|\frac{dx}{dt}\right|_T$$

From here on undamped sinusoidal oscillation according to the equation

$$MH_e x = -I\frac{d^2x}{dt^2}$$

$$x = A \sin 2\pi\frac{t}{T}$$

Inserted

$$MH_e = \left(\frac{2\pi}{T}\right)^2 I$$

$$\left(\frac{dx}{dt}\right)_{t=T} = \frac{2\pi}{T}A$$

$\int idt = \frac{R}{2\pi}H_e\left(\frac{T}{2\pi}\right)^2\cdot\frac{2\pi}{T}A = \frac{RT}{(2\pi)^2}H_e A$, where A is the maximum deflection in absol. angular measure.

According to Deprez $ki = I\frac{d^2x}{dt^2}$ initial period.

$$\int idt = \frac{I}{\kappa}\left\{\frac{dx}{dt}\right\}_T \quad (1)$$

For the process thereafter $\frac{d^2x}{dt^2} = -\frac{\Theta}{I}x \quad A \sin 2\pi\frac{t}{T}$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{\Theta}{I} \quad (2)$$

$$\kappa i_0 = \Theta x_0 \qquad \kappa\Theta^{-1} = \text{susceptibility } \eta \quad (3)$$

$$\int idt = \frac{I}{\kappa}\cdot\frac{2\pi}{T}A = \frac{I}{\Theta\eta}\frac{2\pi}{T}A = \frac{T}{2\pi\eta}A.$$

$$\frac{1}{\kappa} = \frac{1}{\Theta\eta}$$

Thus, when the susceptibility η for the direct current is known, then quantities of electr. can be measured abs. e.g., with Deprez. Damping can also be calc.

[p. 51] Then one has

$$I\frac{d^2x}{dt^2} = -\Theta x - R\frac{dx}{dt}$$

Such lin equations with const. coefficients are most conveniently treated using imaginary quantities

$$e^{j\omega t} = \cos \omega t + j\sin \omega t \qquad e^{(-\alpha + j\omega)t} = e^{-\alpha t}e^{j\omega t} = e^{-\alpha t}(\cos \omega t + j\sin \omega t)$$

Instead of $A\cos \omega t$ and $Ae^{-\alpha t}(\cos \omega t)$, one inserts $Ae^{\gamma t}$, where γ can be compl. The real part of this solution is then the solution sought.
Now, again

$$\int idt = \frac{I}{\kappa}\left\{\frac{dx}{dt}\right\}_T$$

Now, the above equation

$$\frac{d^2x}{dt^2} + \frac{R}{I}\frac{dx}{dt} + \frac{\Theta}{I}x = 0 \qquad e^{\lambda t} \text{ solution}$$

$$\lambda^2 + \frac{R}{I}\lambda + \frac{\Theta}{I} = 0 \qquad \left(\lambda + \frac{1}{2}\frac{R}{I}\right)^2 = \frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{\Theta}{I}$$

$$\lambda = -\frac{1}{2}\frac{R}{I} \pm \sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{\Theta}{I}}$$

Let discriminant be negative $\lambda = -\underbrace{\frac{R}{2I}}_{\alpha} \pm i\underbrace{\sqrt{\frac{\Theta}{I} - \frac{1}{4}\left(\frac{R}{I}\right)^2}}_{\omega}$

$x = Ae^{-\alpha t} \sin \omega t$ solution.

Discussion of the solution. Damped oscillation $\left\{\frac{dx}{dt}\right\}_{t=0} = A\omega$

Oscillation period: $\frac{2\pi}{T} = \omega$

Damping $e^{\alpha T}$ = ratio $\frac{x_1}{x^2}$

Calculation of the first point of reversal

$$\frac{dx}{dt} = A\{-\alpha e^{\cdot\cdot} \sin + \omega e^{\cdot} \cos\} = Ae^{-\alpha t}\sqrt{\alpha^2 + \omega^2}\sin(\varphi - \omega t) \qquad \sin\varphi = \frac{\omega}{\sqrt{}}$$

$$\frac{dx}{dt} = 0 \text{ für } t = \frac{\varphi}{\omega} = \frac{1}{\omega}\operatorname{arctg}\frac{\omega}{\alpha} \qquad \cos\varphi = \frac{\alpha}{\sqrt{}} \quad \operatorname{tg}\varphi = \frac{\omega}{\alpha}$$

$$x_{max} = Ae^{-\alpha(\varphi/\omega)}\sin\varphi, \text{ where } \operatorname{tg}\varphi = \frac{\omega}{\alpha}$$

$$x_{max.} = \frac{1}{\omega}\left|\frac{dx}{dt}\right|_{t=0} e^{\cdot\cdot}\sin\varphi = \frac{1}{\omega}\frac{\kappa}{I}\int i\,dt\cdot e^{\cdot\cdot}\sin\varphi$$

$$= \frac{1}{\omega}\eta\sqrt{\left(\frac{2\pi}{T}\right)^2 + \alpha^2}\,Ee^{-\alpha(\varphi/\omega)}\sin\varphi^{[44]} \qquad \operatorname{arctg}\varphi = \frac{\omega^{[45]}}{\alpha}$$

Thus, we can measure the quantity of electricity of a current impulse absolutely by [p. 52] electromagnetic means. We have seen earlier that the electr. quantities can be measured absolutely by static means. Since voltages can be measured absolutely with Thomson's balance,[46] & the capacities calculated.

Dimensions $\frac{E_s^2}{L} = M\frac{L^2}{T^2}$ $\quad E_s = M^{1/2}L^{3/2}T^{-1}$

$$\mathfrak{H} = M^{1/2}L^{-1/2}T^{-1}$$

$$\frac{i\,ds}{R^2} = \frac{i}{l} = \mathfrak{H} = M^{1/2}L^{-1/2}T^{-1}$$

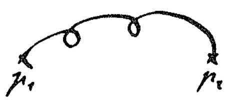
$$i = M^{1/2}L^{+1/2}T^{-1}$$

$$E_m = \int i\,dt = M^{1/2}L^{1/2}$$

$\frac{E_s}{e_m} \approx \frac{L}{T}$ experiment yielded $3\cdot10^{10}$ = velocity of light = c.

This result led to Maxwell's theory of light. Remark. The fact that $\frac{E_s}{E_m}$ is indep. of the experimental design justifies the assumption that i_m is equal to the quantity of static electricity transported per unit time. $\frac{i_s}{i_m} = c$

Unit for Voltage. Ohm's Law



We consider a piece of conductor that is not acted upon by any electromot. forces (Expl.) The electric energy supplied to this piece per second is $p_1 i - p_2 i$ (electrostatically) = $\Delta p_s \cdot i_{st}$.

Effekt erg $= \frac{c\Delta p_s}{\Delta p_m} \cdot i_m = \Delta p_m \cdot i_m$

We have thus obtained a new absolute unit for the voltage. Calorimetrically, if no effect other than heat is produced.

[p. 53] Practical unit constructed, which is 10^8 greater

$$10^{-8}\Delta p_m = \Delta p_{pr} \quad 10 i_m = i_{pr.}$$

$$\text{Effect} = \Delta p_m \cdot i_m = \Delta p_{pr}\cdot 10^7$$

$$\int i_{pr.}dt = E_{pr.} \quad \text{Coul. unit} \quad E_{pr} = 10E_m$$

It turns out that for metallic & electrolytic conductors at constant temperature $\frac{\Delta p}{i}$ is a constant; one calls it the resist w of the conductor.

$$\Delta p = iw. \text{ (Ohm's law)}$$

w depends on geometrical conditions and on constants characteristic of the material. For [large] hom rod[47]

$$w = \omega\frac{l}{1}\ \omega \text{ spec. resist. } \frac{1}{\omega} = \sigma \text{ conductivity of the material.}$$

It is possible to calculate the resist. of solid conductors if the current distr. is known. For a linear current we have

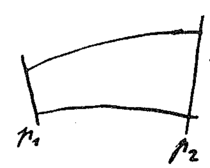
$$\Delta p = i_V \cdot \int_V \frac{\omega dl}{q} \quad (\omega \,\&\, q \text{ are funct. of } l$$

$$i = \sum i_V = \Delta p \sum \frac{1}{\int_V \frac{\omega dl}{q}} = \Delta p \sum \frac{1}{w_V}$$

The practical unit of resistance, the Ohm, is so defined that we get the equation

$\Delta p_{pr.} = i_{pr.}\cdot w_{pr} 10^{-8}\Delta p_m = 10 i_m \cdot w_{pr.} 10^{-9} w_m = w_{pr.}$

Determ. of the current flow in solid conductors.



$$i_x = -\frac{\partial\varphi}{\partial x}\cdot\sigma \qquad \frac{\partial}{\partial x}\left(\sigma\frac{\partial\varphi}{\partial x}\right) + \cdot + \cdot = 0$$

------

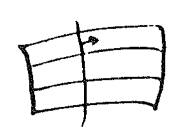
if homogen $\Delta\varphi = 0$

On surface $\quad$ [--] $i_x \cos nx + \cdot + \cdot = 0$

$$\frac{\partial\varphi}{\partial n} = 0$$

Mathematical problem the same as in electrostatics.

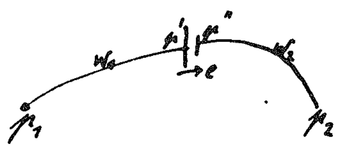
Math. Relationship between Resist. & Capacity[48] [p. 54]



$$i = \sigma\int -\frac{\partial\varphi}{\partial n}d\sigma = \sigma\cdot 4\pi E_s$$

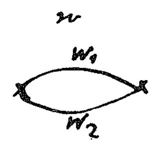
$$w = \frac{\Delta\varphi}{i} = \frac{1}{4\pi\sigma}\frac{\Delta\varphi}{E_s} = \frac{1}{4\pi\sigma}\cdot\frac{1}{C_s} = \frac{\omega}{4\pi}\frac{1}{C_s}$$

The capacity problem and resistance problem are, thus, identical. We give the mater. resistance wherever electrost. cap. has been calculated.
Ohm's Law, if electrom. forces[49]



$$p_1 - p' = iw_1 \quad 1$$
$$p'' - p_2 = iw_2 \quad 1$$
$$\underline{p'' - p' = e \quad -1}$$
$$i(w_1 + w_2) - e = p_1 - p_2$$
$$iw = e + (p_1 - p_2)$$

Also applicable if electromot. forces are uniformly distributed. Special case starting point & end point coincide. Then $e = iw$, if w total resist. of the circuit. Resistances connected in parallel.

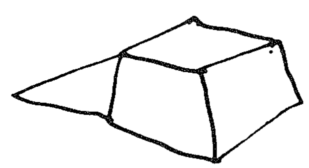


$$\Delta p = i_1 w_1 = i_2 w_2$$

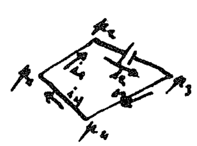
$$i_1 + i_2 = i = \Delta p\left(\frac{1}{w_1} + \frac{1}{w_2}\right) = \frac{\Delta p}{w}$$

$$\frac{1}{w} = \frac{1}{w_1} + \frac{1}{w_2} \;.$$

Kirchhoff's Laws for Current Networks



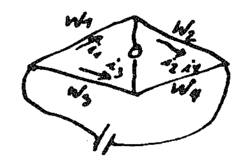
1) In junction $\sum i = 0$, because otherwise incess. accumulation of charge.
2) Any polygon considered



[p. 55]

$$p_2 - p_1 + e_1 = i_1 w_1$$
$$p_3 - p_2 + e_2 = i_2 w_2$$
$$\text{-----------------}$$
$$\underline{p_1 - p_4 + e_4 = i_4 w_4}$$
$$\sum e = \sum iw$$

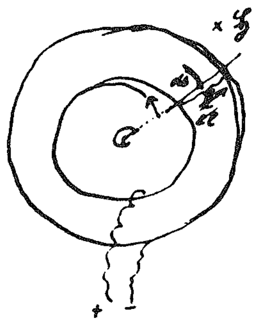
Application to Wheatst. bridge[50]



$$i_2 = i_1 \qquad i_4 = i_3$$
$$i_1 w_1 - i_3 w_3 = 0$$
$$\underline{i_1 w_2 - i_3 w_4 = 0}$$
$$\frac{w_1}{w_3} = \frac{w_2}{w_4} \qquad \frac{w_1}{w_2} = \frac{w_3}{w_4}$$

Solid conductors only here.[51]

Electric Induction [force work on the path ds

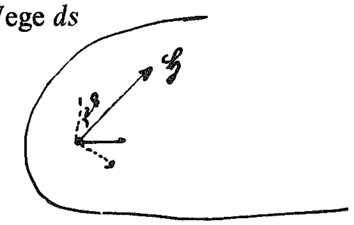


$il\mathfrak{H}$ = Kraft

$il\mathfrak{H}\, ds$ = Arbeit auf dem Wege ds

$= ei\,dt$

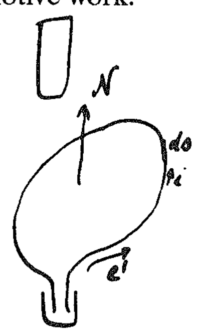
$$e = l\mathfrak{H}\frac{ds}{dt}.$$



Motional Field

If a circuit is displaced in a magn field, the system expends work that is equal to idN, where N is the number of the lines of force traversing the field and the lines of force originating from the current itself are neglected. The result holds also if magnetization constant μ.[52]

[p. 56] If the field originates from magnets and the total energy of the field does not change in the course of the displacement, then an electromagnetic force must counteract the current, a force against which we must apply electrical work that is equal to the ponderomotive work.



$$idN = e'idt = -eidt$$

$$e = -\frac{dN}{dt}$$

If e is measured in practical units, then $e_{\text{volt}} = -10^{-8}\frac{dN}{dt}$.

This is the induction law of Faraday. Since the origin of the magnetic field is obviously not important, the law is generally valid, no matter how the field might be produced.

Extension to the case where the magnet is in motion and the conductor at rest.

Provides a method for the determination of magnetic fields and their changes.

The current field also acts on this current itself, if the current is changed. From this it follows that every linear current can be conceived as a bundle of linear currents. Insofar as one can view N as defined, one has again

$$e = -\frac{dN}{dt}$$

But now one has to set $N = L \cdot i$, where i is the instantaneous current strength, thus also [p. 57]

$$e = -\frac{dL_i}{dt},$$

or, if L is independent of the time:

$$e = -L\frac{di}{dt} \qquad e_{pr.}\cdot 10^8 = -L\ 10^{-1}\frac{di_{pr.}}{dt} \qquad (L\cdot 10^{-9}) = L_{pr.}$$

L is the coefficient of self-induction. The practical unit of self-induction is the Henry = 10^9 abs. The equations are then valid for pr. un. as well.

Solenoid:[53]

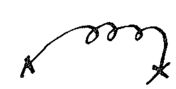
$$N = \frac{4\pi ni}{l}f$$

$$e = -n\frac{dN}{dt} = -\frac{4\pi n^2 f}{l}\frac{di}{dt}$$

L

Ring analogous. If permeability μ, then $L\mu$ times greater

Linear conductor through which variable current flows.



$$\Delta p + e = iw$$

$$-L\frac{di}{dt}.$$

If self-ind. the only electromot. force, then

$$\Delta p = iw + L\frac{di}{dt}$$

Conductor in zero-current state suddenly connected to potential difference.

How does current increase? $P = iw + L\dfrac{di}{dt}$

$$i = \frac{P}{w} + i_1$$

$$i_1 w = -L\frac{di}{dt} \quad \frac{d\lg i}{dt} = -\frac{w}{L} \quad i = \text{const } e^{-(w/L)t}.$$

$$i = \frac{P}{w}\left(1 - e^{-(w/L)t}\right)$$

Time: $T\frac{w}{L} = 5 \quad T = \frac{5L}{w}$ Practically very short time.

Fading away of the current analogous.

[p. 58] Sine current

$$\Delta p = iw + L\frac{di}{dt}$$

$\Delta p = A \cos \omega t$ given

$i = B \cos(\omega t - \varphi)$ — φ is then the phase difference between voltage & current. If φ pos, then current lags.

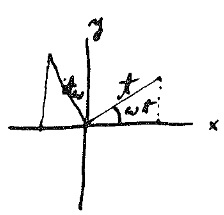
$$B(\underbrace{w\cos(\omega t - \varphi)}_{w\cos\alpha} - \underbrace{\omega L}_{\sin\alpha}\sin(\omega t - \varphi)) \qquad \operatorname{tg}\alpha = \frac{\omega L}{w}$$

$$A\cos\omega t = B\sqrt{w^2 + \omega^2 L^2}\cos(\omega t - \varphi + \alpha)$$

$$B = \frac{A}{\sqrt{w^2 + \omega^2 L^2}} \qquad \varphi = \alpha$$

Graphic Illustration with Rotating Vectors

$A \cos \omega t$

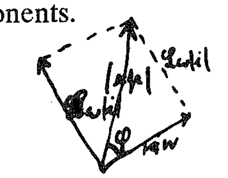


Derivative

$-A \cos \omega t$

Thus, rule for differentiation.

Rule for summing parallelogram, because projection of the resultant always sum of projections of the components.



$$\operatorname{tg}\varphi = \frac{\omega L}{w} \qquad P^2 = i^2(w^2 + (\omega L)^2)$$

Comes to the same as the replacement of the trigonometric funct by exp. with complex [p. 59] arg.[55] $A\cos(\omega t - \varphi)$ is the real portion of $Ae^{j(\omega t - \varphi)} = Ae^{-j\varphi}e^{j\omega t}$, $= \mathfrak{A}e^{j\omega t}$, where $\mathfrak{A}$ complex $= Ae^{-j\varphi}$. Thus, phase angle & amplitude known if $\mathfrak{A}$ known.

$\Delta p = \mathfrak{P}e^{j\omega t}$	$\Delta p = Pe^{j\omega t}$
$i = \mathfrak{J}e^{j\omega t}$	$i = Ie^{j(\omega t - \varphi)}$
$\mathfrak{P} = \mathfrak{J}w + j\omega L\mathfrak{J} = I(w + j\omega L)$	$P = Iwe^{-j\varphi} + j\omega LIe^{-j\varphi}$
$\frac{\mathfrak{P}}{\mathfrak{J}} = \frac{Pe^{-\varphi_p}}{Ie^{-\varphi_i}} = \underbrace{(w + j\omega L)}_{\sqrt{w^2 + (\omega L)^2}\,e^{j\,\mathrm{arctg}\frac{\omega L}{w}}}$	$= I(w + j\omega L)e^{j\varphi}$
$\frac{P}{I} = \sqrt{w^2 + (\omega L)^2} \quad \varphi_i - \varphi_p = \mathrm{arctg}\frac{\omega L}{w}$	etc.

Calculation considerably simpler than with sin & cos. Therefore almost always applied nowadays.

The calculation is simplest if by variables one immediately understands corresp. compl. Then we get at once

$$\Delta p = i(w + j\omega L)$$

i-vector to be multiplied by $(w + j\omega L)$ vector in order to have Δp. Coincides with theory of rotating vectors. Naturally, these methods are applicable only to harmonic functions.

Earth Inductor—Measurement of Self-Induction

Magnetic energy of a circuit

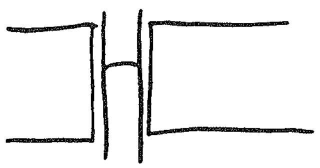
$$\int e' \, idt = -\int eidt = \int \frac{dN}{dt}idt = L\frac{i^2}{2}.$$

[p. 60]

Remark about Ponderomotive Effects on Magnetizable Bodies in the Field of a Current[56]

$$dA = e'\, i dt = -e i dt = \frac{dN}{dt} i dt$$

The bodies seek to move in such a way that N becomes a maximum. This fact forms the basis for the measurement of small <dielectric> magnetization constants.[57]



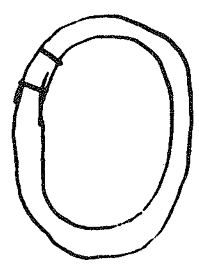
Homogeneous field $\mathfrak{H}$ in the body $H\mu$
If column rises by δh, then the change in the energy is

$$\frac{1}{8\pi}\mu\mathfrak{H}^2 q\,\delta h - \frac{1}{8\pi}\mathfrak{H}^2 q\,\delta h.$$

Thus force on column $= \frac{\mu - \mu_0}{8\pi}\mathfrak{H}^2 q \;\Big|\; = qh\rho g,$ where h is the height of the rise produced by the magnetic force.

Energy & Energy Principle

Previously: For a circuit we have



$$E = \frac{1}{8\pi}\int \mathfrak{H}\mathfrak{B}\, \underset{q dl}{dt} = \frac{1}{8\pi}\int \mathfrak{H}\, dl dN = \frac{1}{2} iN$$

Since by definition $N = Li$, one obtains $E = \frac{Li^2}{2}$ in agreement with the above analysis.

Application of the energy principle to current of constant intensity.

$$e'\, i dt = dE + dA$$

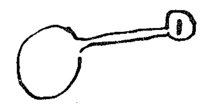
$$+\, i\frac{dN}{dt} = \frac{1}{2}\frac{d(iN)}{dt} + dA$$

Thus, $$dA = \frac{1}{2} i \frac{dN}{dt} - \frac{1}{2} N \frac{di}{dt} = \frac{1}{2} i \frac{d(Li)}{dt} - \frac{1}{2} iL \frac{di}{dt}$$

If i is const., we obtain $dA = \frac{1}{2} idN = dE$ [p. 61]

Work is equal to the increase in energy. The expression differs from that for the work of the current in an external magnetic field by the factor $\frac{1}{2}$. Example. Parallel currents.

Measurement of an EMF of Short Duration. Earth Inductor[58]



$$e = iw - L\frac{di}{dt}$$

At the start $i = 0$. At the end $i = 0$

$$\int edt = w\int idt - \underset{\substack{\parallel \\ 0}}{L\,|i|_0^t}$$

For the earth inductor $e = n\frac{dN}{dt}$ $\int edt = 2Nn.$

Quantity of electricity measured with ballistic instrument. Analogous method for the investigation of hysteresis.

Interaction between Permanent Magnets & Current

N_m = circuit-traversing flux that originates from the magnet
N_i = " " " " " " " current. L self-ind.

$$dA = idN_m + \frac{1}{2}i^2 dL$$

$$e = -\frac{dN_m}{dt} - \frac{d(iL)}{dt}$$

Ohm's equation
$e' + e = iw$

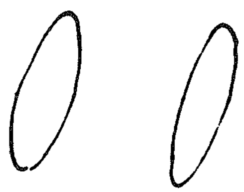
Energy principle $i^2 w\,dt = -i\,dN_m - i\,diL + e'i\,dt$

$$= -dA + \underbrace{\frac{1}{2}i^2\,dL - id(iL)}_{-\frac{1}{2}i^2\,dL + \frac{1}{2}L\,di^2} + e'i\,dt$$

$$= -dA + e'i\,dt + d\left(\frac{1}{2}Li^2\right)^{[59]}$$

[p. 62] *Interaction between Two Circuits*

The circuits are immobile.



L_1 = flux that current 1 of strength 1 yields through its surface
M_{12} = " " " 1 " " " " boundary of cur. 2
M_{21} = " " 2 " " " " " cur. 1
L_2 = " " " 2 " " " " its boundary.

Total flux through 1): $L_1 i_1 + M_{21} i_2 = N_1$
" " " 2): $M_{12} i_1 + L_2 i_2 = N_2$

The equation for the two circuits is

$$e_1 - \frac{dN_1}{dt} = i_1 w_1$$

$$e_2 - \frac{dN_2}{dt} = i_2 w_2$$

or

$$e_1 = i_1 w_1 + L_1\frac{di_1}{dt} + M_{21}\frac{di_2}{dt}$$

$$e_2 = i_2 w_2 + M_{12}\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

What form does the energy principle take?[60]

$$e_1 i_1\,dt = i_1^2 w\,dt + \frac{d}{dt}\left(\frac{1}{2}L_1 i_1^2\right) + M_{21} i_1 \frac{di_2}{dt}dt$$ [p. 63]

$$e_2 i_2\,dt = i_2^2 w\,dt + M_{12} i_2 \frac{di_1}{dt}dt + \frac{d}{dt}\left(\frac{1}{2}L_2 i_2^2\right)$$

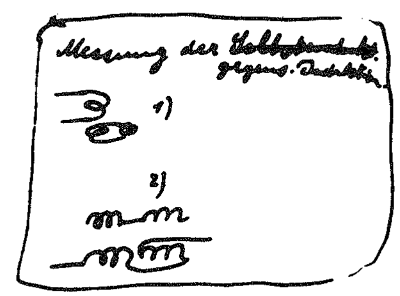
$$dA_e = G + dE_{m1} + dE_{m2} + M_{21} i_1\,di_2 + M_{12} i_2\,di_1$$

$dA_e - G$ must be tot. differential. Thus $M_{21} = M_{12} = M$.

$E = \frac{1}{2}(L_1 i_1^2 + 2M i_1 i_2 + L_2 i_2^2)$ must never be negative

$L_1 + 2Mx + L_2 x^2 | M + L_2 x = 0$ $\quad L_1 - 2\frac{M^2}{L_2} + \frac{M^2}{L_2} > 0$

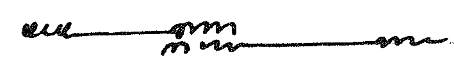
Measurement of mutual induction.



One can also resolve the magnetic field in another way.

Φ number of lines of force traversing both circuits.
Φ_1 " " " " " " only 1)
Φ_2 " " " " " " " 2).

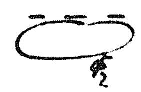
Model



This resolution is especially advantageous when almost all lines of force traverse both circuits. How do we determ Φ_1, Φ_2 & Φ?

$$\Phi_1 = \frac{N_1}{n_1} = \frac{L_1}{n_1} i_1 + \frac{M}{n_1} i_2 = \underbrace{\left(\frac{L}{n_1} - \frac{M}{n_2}\right)}_{\Phi_1} i_1 + M \overbrace{\left(\frac{i_2}{n_1} + \frac{i_1}{n_2}\right)}^{\Phi}$$

Φ_2 -------------- = ----------------------

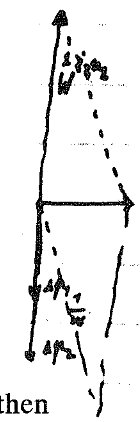


Transformer, with Resist. & Leakage Neglected [61]

$$\Delta p_1 = n_1 \frac{d\Phi}{dt}$$

$$\Phi = \frac{1}{w}(i_1 n_1 + i_2 n_2)$$

$$\Delta p_2 = n_2 \frac{d\Phi}{dt}$$



The phase of the current depends on what is switched on. If only resist. then
[p. 64] i_2 phase of Δp_2.

Two mobile circuits
Work el. force. Energy

$$p_1 = i_1 w_1 + \frac{dL_1 i_1}{dt} + \frac{dM i_2}{dt}$$

$$p_2 = i_2 w_2 + \frac{dM i_1}{dt} + \frac{dL_2 i_2}{dt}$$

$$d'A_1 = \frac{1}{2} i_1 d(L_1 i_1) + i_1 d(M i_2)$$

$$d'A_2 = \frac{1}{2} i_2 d(L_2 i_2) + i_2 d(M i_1)$$

$$d'A_{e1} = p_1 i_1\, dt$$

$$d'A_{e2} = p_2 i_2\, dt$$

$$E_m = \frac{1}{2}(L_1 i_1^2 + 2M i_1 i_2 + L_2 i_2^2)$$

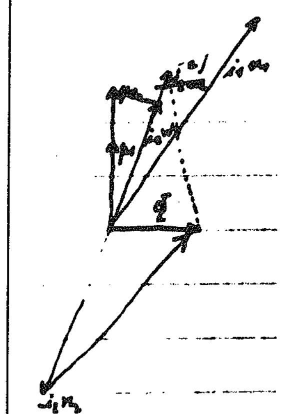
$d'A_e = d'G + dE_m + d'A_e$ [62]

$$p_1 = n_1 \frac{d\Phi}{dt}$$

$$p_2 = n_2 \frac{d\Phi}{dt}$$

$$p_2 = i_2 w_2 + L_2 \frac{di_2}{dt}$$

$$\Phi = \frac{1}{w}(i_1 n_1 + i_2 n_2)$$



Transformer with imaginary.[63] By $e_1 i_1$ . . . one immediately understands imaginary vectors.

$$\begin{array}{ll|l|l} e_1 = i_1(w_1 + j\omega L_1) & + i_2 \cdot j\omega M & w_2 + j\omega L_2 & j\omega M \\ 0 = i_1 j\omega M & + i_2(w_2 + j\omega L_2) & -j\omega M & -(w_1 + j\omega L_1) \end{array}$$

$$e_1 = i_1 \frac{[(w_1 + j\omega L_1)(w_2 + j\omega L_2) + \omega^2 M^2]}{(w_2 + j\omega L_2)}$$

$$= i_1 \left[w_1 + j\omega L_1 - \frac{\omega^2 M^2}{w_2 + j\omega L_2} \right]^{[64]}$$

$$e_1 = i_2 \frac{-(w_1 + j\omega L_1)(w_2 + j\omega L_2) - \omega^2 M^2}{j\omega M}$$

In the absence of leakage ($L_1 L_2 - M_2 = 0$),[65] the second equation becomes

$$e_1 = -i_2 \frac{w_1 w_2 + j\omega(L_1 w_2 + L_2 w_1)}{j\omega M}$$

& if, in addition, $w_1 = 0$, then $e_1 = -i_2 \frac{L_1 w_2}{M} = -i_2 w_2 \frac{n_1}{n_2} \quad \frac{n_1^2}{n_1 n_2}$

$$\text{The [--] in } e_1 = i_1 \frac{j\omega L_1 w_2}{w_2 + j\omega L_2} = i_1 \left(w_2 \frac{n_1^2}{n_2^2} \right)$$

or if w_2 negligible compared with $j\omega L_2$

[p. 65] *Capacity*

$E_m = C_m p_m$ If this equation is to be valid in electromagnetic units, then the unit of capacity is fixed thereby. What is the relation between this unit and the static unit?

$$E_m = \frac{1}{c} \cdot E_s$$

$$p_m = c \cdot p_s$$

$$\text{hence } \frac{1}{c} E_s = C_m c p_s$$

$$E_s = \underbrace{(C_m c^2)}_{C_s} p_s \qquad C_{st} = C_m \cdot c^2$$

The static unit is c^2 times smaller than the electromagnetic one
There is also a practical unit

$$E_{pr} = C_{pr} p_{pr}.$$

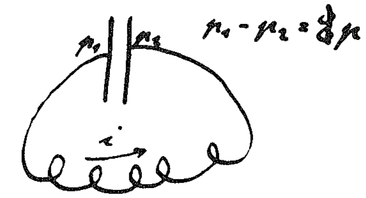
$$10 E_m = C_{pr.}\, 10^{-8} p_m \qquad E_m = \underbrace{(10^{-9} C_{pr.})}_{C_m} p_m \qquad \underbrace{C_{pr.} = 10^9 C_m}_{\text{Farad}}$$

Practical unit 10^{-9} of the absolute magnet. unit
This is $9 \cdot 10^{20}$ electrostatic units
Practical unit (farad) $9 \cdot 10^{11}$ electrostatic units In addition, microfarad 10^{-6} of the farad. $9 \cdot 10^5$ electrostatic units.

Circuit with Capacitance and Self-Induction. Electric Oscillations

$$pC = E \qquad -\frac{dE}{dt} = -C\frac{dp}{dt} = i$$

$$p - L\frac{di}{dt} = iw$$



differentiated once again
[if...then..or solution; oscillation period easily realizable]

$$\frac{dp}{dt} = \frac{di}{dt} w + L\frac{d^2 i}{dt^2} = -\frac{i}{C}$$

$$\text{also } \frac{1}{C} i + w\frac{di}{dt} + L\frac{d^2 i}{dt^2} = 0$$

$$i + wC\frac{di}{dt} + LC\frac{d^2 i}{dt^2} = 0$$

Wenn $w = 0$, dann $\mathfrak{J} \cos \omega t$ oder $I e^{j\omega t}$ Lösung.

$$\mathfrak{J} + (j\omega)^2 LC \mathfrak{J} = 0 \qquad \omega = 2\pi n = \sqrt{\frac{1}{CL}}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{1}{CL}}$$

$$\begin{array}{ccc} \omega 10^{[2]} & 10^{[+6]} & 10^{[+8]} \\ 10^8 & \omega 10^{-10} & 10^{-8} \end{array}$$

Schwingungsdau 10^{-4} Sek. wohl realisierbar,

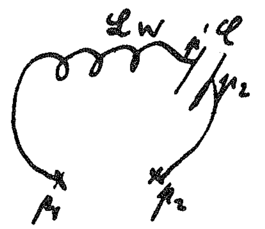
$$1 + \alpha w C + \alpha^2 LC = 0$$

$$\alpha^2 + \alpha\frac{w}{L} + \frac{1}{CL} = 0$$

$$\alpha = -\frac{w}{2L} \pm \sqrt{-\frac{1}{LC} + \left(\frac{w}{2L}\right)^2}$$

[p. 66] Frequency somewhat influenced (reduced) by resistance. Amplitude decreases with $e^{-\frac{W}{2L}t}$ $\left(W = 1 \text{ ohm} \ \& \ L = \frac{1}{100} \ T = \frac{1}{50}.\right)$

We shall also discuss in particular the case of sinusoidal currents.



$$p_1 - p' = iw + L\frac{di}{dt}$$

$$p' - p_2 = \frac{1}{C}E = \frac{1}{C}\int i\,dt$$

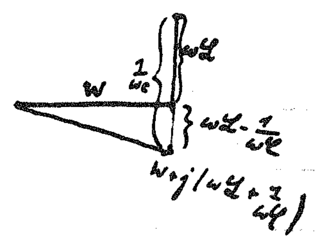
$$p = p_1 - p_2 = iw + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$$

Solution by means of imag. $i = \mathfrak{J}_0 e^{j\omega t}$ $\int idt = \frac{I_0}{j\omega}e^{j\omega t} = \frac{i}{j\omega}$

Inserting this, one obtains

$$p = i\left(w + j\omega L + \frac{1}{j\omega C}\right) = i\left(w + j\left(\omega L - \frac{1}{\omega C}\right)\right)$$

$$-j\frac{1}{wC}$$



$$\mathfrak{P} = \mathfrak{J}\sqrt{w^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{j\varphi} \qquad \operatorname{tg}\varphi = \frac{\omega L - \frac{1}{wC}}{\omega}$$

If $i = I\cos\omega t$

$$p = I\sqrt{\quad}\cos(\omega t + \varphi)$$

$$\text{Ampl. } I = \frac{P}{\sqrt{w^2 + \left(\omega L - \frac{1}{\omega c}\right)^2}}$$

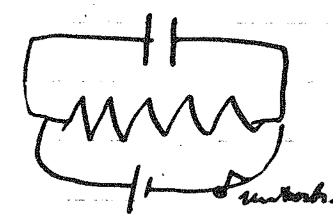
Resonance when I maximum $\omega = \frac{1}{\sqrt{cL}}$ proper oscillations. For the latter, I becomes infinite when $w = 0$ at given voltage. Capacity compensates self-ind. But only for specif. perm. With very weak terminal voltage considerable current.

When there is resonance, voltage on the condenser $\frac{1}{C}\int idt = p' - p_2 = \frac{P}{Cw}\cos.$[66] [p. 67]

May become enormously large if C small & W small.

Energy Principle in Oscillations

$$p = iw + L\frac{di}{dt}\Big|\, i\,dt$$



$$pi\,dt = i^2 w\,dt + d\left(\frac{L}{2}i^2\right)$$

$$-c\frac{dp}{dt}$$

$$\text{If } w = 0 \quad \frac{d}{dt}\left(c\frac{p^2}{2} + \frac{L}{2}i^2\right) = 0$$

The to-and-fro oscillation of energy[67] $p_m^2 = \frac{L}{c}i_m^2$

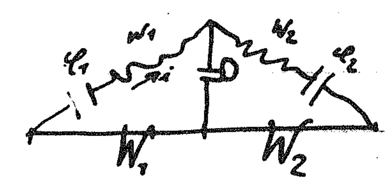
$$p_m = i_m\sqrt{\frac{L}{c}} \qquad \text{If} \quad L = 10^{-2} \text{ Henry}$$

$$C = 10^{-8} \text{ Farad}$$

$$p_m = 10^3\, i_m$$

Comparison of capacities[68]

$$\frac{p_1}{p_2} = \frac{W_1}{W_2} = \frac{i\left(w_1 - j\frac{1}{wC_1}\right)}{i\left(w_2 - j\frac{1}{wC_2}\right)}$$



From this the relationship (independent of period).
Rapid oscillations if L small. Not coils but simple wires.
Back-and-forth loop[69]

$$\int_0^{R_1} 2i\frac{r^2}{R^2}\cdot\frac{1}{r}\,dr = i$$

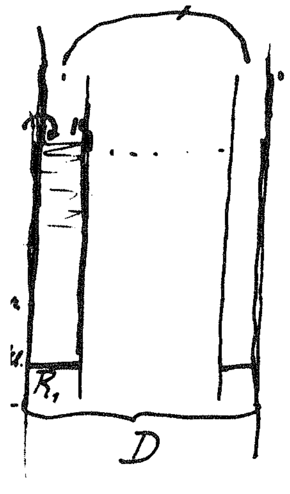
$$\int_{R_1}^{D} \frac{2i}{r}\,dr = 2i\lg\frac{D}{R_1}$$

Thus, all in all

$2 + 2\lg\frac{D^2}{R_1R_2}$ too large

$2\lg\frac{D^2}{R_1R_2}$ too small

$1 + 2\lg\frac{D^2}{R_1R_2}$



[p. 68]

$$L = l\left(1 + 2\lg\frac{D^2}{R_1R_2}\right)$$

If we introduce the total length $l' = 2l$, & set $R_1 = R_2$, then

$$L = l'\left(\frac{1}{2} + 2\lg\frac{D}{R}\right)$$

We obtain the approximate value of L for a square.

$$L = 2l'\left(\frac{1}{4} + \lg\frac{s}{R}\right) = 2l'\left(\lg\frac{l'}{R} - 1.13\right)^{[70]}$$

$$\lg\frac{l'}{4R}$$

Is too large, because field calculated too large. In reality, according to rigorous calculation[71] $L = 2l'\left(\lg\frac{l'}{R} - 1.9\right)$.

For circle the same formula but -1.5.

Waves in a Wire (Distributed Capacity)

c = capacity per unit length.
p pot. e el. quant. " " " i current str.

$$-\frac{\partial i}{\partial x} = c\frac{\partial p}{\partial t}$$ (continuity equation for electricity

$$iw + l\frac{\partial i}{\partial t} = -\frac{\partial p}{\partial x}^{[72]} \quad \Big| \; c\frac{\partial}{\partial t}$$

These are differential equations for i & p.
p eliminated

$$cw\frac{\partial i}{\partial t} + c\ell\frac{\partial^2 i}{\partial t^2} = \frac{\partial^2 i}{\partial x^2}$$

p can then be determined from the first equation. If w neglected, then

$$cl\frac{\partial^2 i}{\partial t^2} = \frac{\partial^2 i}{\partial x^2} \qquad i = f(x - Vt) \text{ is solution}$$

$$clV^2 = 1$$

$$V = \frac{1}{\sqrt{cl}}$$

Two parallel wires whose radius is negligible compared with the distance between them[73] [p. 69]

$$l = 2\lg\frac{D}{R}$$

$$\text{cap} = \frac{1}{2\lg\frac{D}{R}}\cdot\frac{1}{c^2}$$

$V = c$ Such electric waves propagate with the speed of light. For other c & l different results.

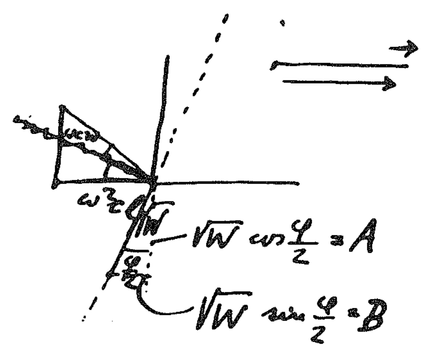
W not neglected. ∞ long wire. Sinusoidal solution. Influence of ε & μ.

$i = Xe^{j\omega t}$

$\underbrace{(j\omega cw - \omega^2 cl)}X = X''$ $\qquad X = Ae^{\gamma x}$

$\gamma^2 = -\omega^2 cl + j\omega cw \qquad \sqrt{(\omega^2 cl)^2 + (\omega cw)^2} = W$

$\gamma = \sqrt{}$



The solution is

$i = Ae^{j\omega t}e^{(-Aj-B)x}$

$= Ae^{-Bx}e^{j\omega(t-(A/\omega)x)} \qquad \frac{\omega}{A}$ velocity

B = damping constant.

A = current amplitude at start.

$W = \omega c\sqrt{w^2+\omega^2 l^2} \qquad \frac{\omega}{A} \sim \frac{\omega}{\sqrt{w}} \sim \frac{1}{\sqrt{cl}}$

$\mathrm{tg}\,\varphi = \frac{w}{\omega t}$

$W = \sqrt{(\omega^2 cl)^2 + (\omega cw)^2}$

$B = \sqrt{W}\ \sin\frac{\varphi}{2}$

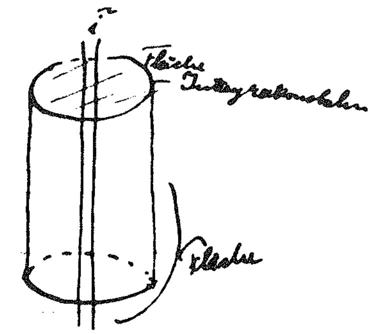
$A = \sqrt{W}\ \cos\frac{\varphi}{2}$

Damping coefficient $B \propto \omega\sqrt{cl}\ \frac{w}{2\omega l} = \frac{w}{2}\sqrt{\frac{c}{l}}$ From this, telephone transmission range.

Pupin's system.[74] | Extreme $w \gg \omega l$.

[p. 70] *Maxwell's Equations*

1) We have reason to assume finite propagation. Conduction with distributed capacity.



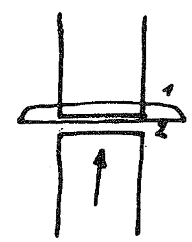
$4\pi i = \int \mathfrak{H}\, ds$ can then no longer be valid for arbitrary surfaces. The law can therefore be strictly maintained only for surface *elements*

$$4\pi j_x = \frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z}$$

----

is therefore surely more exact than the above law in integral form if the currents in question are not constant.

2) Open currents



Conductor interrupted by a dielectric of arbitrary dielectric constant. Condenser. There also seem to be exceptions to $4\pi i = \int \mathfrak{H}\, ds$ for slow currents if one places the surface across the intervening space. This would apply to any intervening space, no matter how narrow. But we can maintain the law in general if we assume that the temporal change of the dielectric associated with the current acts magnetically like a conduction current.

$$\eta = \frac{1}{4\pi}\mathfrak{D}$$

$$E = \int \eta\, d\sigma = \frac{1}{4\pi}\int \mathfrak{D}\, d\sigma$$

$$i = \frac{dE}{dt} = \frac{1}{4\pi}\int \frac{d\mathfrak{D}_n}{dt}\, d\sigma$$

We assume that the right side is equivalent to a current. The X-component of this vector: [p. 71]

$$\frac{1}{4\pi}\frac{\partial \mathfrak{D}_x}{\partial t}$$

Acts like the x-component of a current density (displacement current) Conduction current & displacement current can be present together.

$j_x + \frac{1}{4\pi}\frac{\partial \mathfrak{D}_k}{\partial t} = X$ component of the total current).

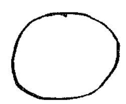
If one corrects the above differential equations in this manner, one obtains

$$4\pi j_x + \frac{\partial \mathfrak{D}_x}{\partial t} = \frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z}$$

$$4\pi j_y + \frac{\partial \mathfrak{D}_y}{\partial t} = \frac{\partial \mathfrak{H}_x}{\partial z} - \frac{\partial \mathfrak{H}_z}{\partial x} \quad \text{in vector not.} \quad 4\pi j + \frac{d\mathfrak{D}}{dt} = \text{curl } \mathfrak{H}$$

$$4\pi j_z + \frac{\partial \mathfrak{D}_z}{\partial t} = \frac{\partial \mathfrak{H}_y}{\partial x} - \frac{\partial \mathfrak{H}_x}{\partial y}$$

These equations are joined by a fourth one, that of Gauss's theorem



$$4\pi E = \int \mathfrak{D}_n d\sigma$$

$$4\pi\rho = \frac{\partial \mathfrak{D}_x}{\partial x} + \frac{\partial \mathfrak{D}_y}{\partial y} + \frac{\partial \mathfrak{D}_z}{\partial z} \qquad 4\pi\rho = \text{div } \mathfrak{D}$$

If one bears in mind that $\frac{\partial \rho}{\partial t} = -\left(\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}\right) = -\text{div } j$, then one has 4π div $j + \frac{\partial}{\partial t}$ (div $\mathfrak{D}$) = 0.

But this equation is contained in the ones above, as one can see by differentiating with respect to x,y,z and adding.

As usual, it is assumed that j and $\mathfrak{D}$ are determined by $\mathfrak{E}$. The simplest hypothesis is

$$j_x = \sigma\mathfrak{E}_x \qquad \mathfrak{D}_x = \varepsilon\mathfrak{E}_x$$

------- --------

------- --------

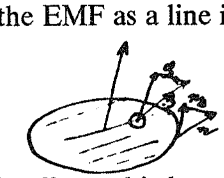
However, the relation can be a more complicated one.

[p. 72] 3) This was the law that defined the magnetic fields determined by electric currents. We have also become acquainted with a law for the production of electromotive effects by the alteration of magnetic fields.

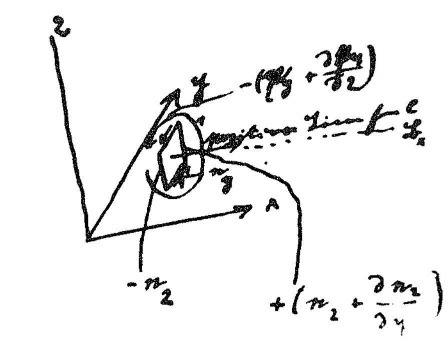
$$e = -\frac{\partial N}{\partial t}$$

This holds first of all for closed circuits. If we think of the EMF as a line integral of an EMF field $\mathfrak{e}$, then the law takes the form

$$\int \mathfrak{e}_s ds = -\frac{d}{dt}\int \mathfrak{B}_n d\sigma$$



Because of the finite propagation velocity of electric effects, this law, too, will only hold for ∞ *small* surface elements. We apply it to the following surface.



$$-\frac{\partial \mathfrak{B}_x}{\partial t} = \frac{\partial \mathfrak{e}_z}{\partial y} - \frac{\partial \mathfrak{e}_y}{\partial z}$$

$$-\frac{\partial \mathfrak{B}_y}{\partial t} = \text{-------}$$

----------------.

Besides this electromotive field we also have an *electric* field $\mathfrak{E}_x\mathfrak{E}_y\mathfrak{E}_z$. This has been taken over from electrostatics. We shall therefore call it $\mathfrak{E}_{sx}$ etc. The following equations hold for it

$$0 = \frac{\partial \mathfrak{E}_{sz}}{\partial y} - \frac{\partial \mathfrak{E}_{sy}}{\partial z}$$

Electromotive & electrostatic field are both def. by the force exerted on the el. unit. We [p. 73] have therefore no reason a priori to consider them as being of different nature. The formal laws also require that the sum $\mathfrak{e}_x + \mathfrak{E}_{sx}$ . . . be considered simply as the elec. field str. $\mathfrak{E}_x$ . . . For if one adds these equations, one obt. [or]

$$-\frac{\partial \mathfrak{B}_x}{\partial t} = \frac{\partial \mathfrak{E}_z}{\partial y}{}' - \frac{\partial \mathfrak{E}_y}{\partial z}$$

----------------- or $-\frac{\partial \mathfrak{B}}{\partial t} = \text{curl } \mathfrak{E}.$

These equations give $\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{B}_x}{\partial x} + \cdot + \cdot\right) = 0.$ Thus, they are compatible with the condition div $\mathfrak{B} = 0$ (There is no real magnetism).

Plane waves.

Let $\mathfrak{B} = \mu\mathfrak{H}$ & $\mathfrak{D} = \varepsilon\mathfrak{E}$, & let μ & ε be indep. of the location. Then the equations read:

We are looking for waves propagating in the X-direction. Everything dep. only on x & t. Let <$F(x - vt)$> be the dependence of all components

$$-\frac{\mu}{c}\frac{\partial \mathfrak{H}_x}{\partial t} = 0 \qquad \frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_x}{\partial t} = 0$$

$$-\frac{\mu}{c}\frac{\partial \mathfrak{H}_y}{\partial t} = -\frac{\partial \mathfrak{E}_z}{\partial x} \qquad \frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_y}{\partial t} = -\frac{\partial \mathfrak{H}_z}{\partial x}$$

$$-\frac{\mu}{c}\frac{\partial \mathfrak{H}_z}{\partial t} = \frac{\partial \mathfrak{E}_y}{\partial x} \qquad \frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_z}{\partial t} = \frac{\partial \mathfrak{H}_y}{\partial x}$$

If at one location $\mathfrak{H}_x = \mathfrak{E}_x = 0$ initially, then it will also be so in the future
Diagonal pairs mutually independent.

$$\begin{array}{l|l} -\frac{\mu}{c}\frac{\partial \mathfrak{H}_z}{\partial t} = \frac{\partial \mathfrak{E}_y}{\partial x} & \frac{\partial}{\partial x} \\ \frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_y}{\partial t} = -\frac{\partial \mathfrak{H}_z}{\partial x} & -\frac{\mu}{c}\frac{\partial}{\partial t} \end{array} \qquad \frac{\partial^2 \mathfrak{E}_y}{\partial x^2} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2 \mathfrak{E}_y}{\partial t^2} = 0$$

$$\mathfrak{E}_y = \mathfrak{F}(x - vt) \qquad \mathfrak{H}_z = \alpha\mathfrak{F}(x - vt)$$

$$v = \frac{c}{\sqrt{\varepsilon\mu}} \qquad \frac{\mu}{c}\alpha\frac{c}{\sqrt{\varepsilon\mu}}\mathfrak{F} = \mathfrak{F}$$

$$\alpha = \sqrt{\frac{\varepsilon}{\mu}}$$

[p. 74] In the case of vacuum $v = c$ $\mathfrak{H}_z = \mathfrak{E}_y$
For a dielectric $\mathfrak{H}_z\sqrt{\mu} = \mathfrak{E}_y\sqrt{\varepsilon}$, further $v = \frac{c}{\sqrt{\varepsilon\mu}}$
For light waves $\mu = 1$ $v = \frac{c}{\sqrt{\varepsilon}} = \frac{c}{n}$ $n = \sqrt{\varepsilon}$ holds for the majority of simple gases and for some liquids. In general, more complicated relations because the connection between $\mathfrak{D}$ and $\mathfrak{E}$ not so simple. On the generation of electric waves later on.
General differential equation of wave propagation in transparent media

$$\begin{array}{l|l} -\frac{\mu}{c}\frac{\partial \mathfrak{H}_y}{\partial t} = \frac{\partial \mathfrak{E}_x}{\partial z} - \frac{\partial \mathfrak{E}_z}{\partial x} & \frac{\partial}{\partial z} \\ -\frac{\mu}{c}\frac{\partial \mathfrak{H}_z}{\partial t} = \frac{\partial \mathfrak{E}_y}{\partial x} - \frac{\partial \mathfrak{E}_x}{\partial y} & -\frac{\partial}{\partial y} \end{array}$$

$$\underbrace{\frac{\mu}{c}\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z}\right)}_{\frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_x}{\partial t}} = -\frac{\partial}{\partial x}\left(\frac{\partial \mathfrak{E}_x}{\partial x} + \frac{\partial \mathfrak{E}_y}{\partial y} + \frac{\partial \mathfrak{E}_z}{\partial z}\right) + \Delta\mathfrak{E}_x$$

$$\frac{\mu\varepsilon}{c^2}\frac{\partial^2 \mathfrak{E}_x}{\partial t^2} - \Delta\mathfrak{E}_x = 0. \text{ etc}$$

These are the fundamental equations of the wave theory.

The Energy Principle and the Law of Conservation of Momentum

$$\begin{array}{l|c|c} -\frac{\mu}{c}\frac{\partial \mathfrak{H}_x}{\partial t} = \frac{\partial \mathfrak{E}_z}{\partial y} - \frac{\partial \mathfrak{E}_y}{\partial z} & -c\frac{\mathfrak{H}_x}{4\pi} & \\ -\frac{\mu}{c}\frac{\partial \mathfrak{H}_y}{\partial t} = \frac{\partial \mathfrak{E}_x}{\partial z} - \frac{\partial \mathfrak{E}_z}{\partial x} & -c\frac{\mathfrak{H}_y}{4\pi} & \frac{\mathfrak{E}_z}{4\pi} \\ -\frac{\mu}{c}\frac{\partial \mathfrak{H}_z}{\partial t} = \frac{\partial \mathfrak{E}_y}{\partial x} - \frac{\partial \mathfrak{E}_x}{\partial y} & -c\frac{\mathfrak{H}_z}{4\pi} & -\frac{\mathfrak{E}_y}{4\pi} \end{array}$$

$$\begin{array}{l|c|c} 4\pi j_x + \frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_x}{\partial t} = \frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z} & c\frac{\mathfrak{E}_x}{4\pi} & \\ 4\pi j_y + \frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_y}{\partial t} = \frac{\partial \mathfrak{H}_x}{\partial z} - \frac{\partial \mathfrak{H}_z}{\partial x} & c\frac{\mathfrak{E}_y}{4\pi} & \frac{\mathfrak{H}_z}{4\pi} \\ 4\pi j_z + \frac{\varepsilon}{c}\frac{\partial \mathfrak{E}_z}{\partial t} = \frac{\partial \mathfrak{H}_y}{\partial x} - \frac{\partial \mathfrak{H}_x}{\partial y} & c\frac{\mathfrak{E}_z}{4\pi} & -\frac{\mathfrak{H}_y}{4\pi} \end{array}$$

$$\frac{\partial P_E}{\partial t} + c(\mathfrak{E}_x j_x + \mathfrak{E}_y j_y + \cdot) = -\frac{c}{4\pi}\left\{\frac{\partial}{\partial x}(\mathfrak{E}_y\mathfrak{H}_z - \mathfrak{E}_z\mathfrak{H}_y) + \cdot + \cdot\right\}$$ [75]

Is canceled by what comes from the right-hand side of the second system.

$$\pm\frac{c}{4\pi}\frac{\partial}{\partial x}(\mathfrak{E}_y\mathfrak{H}_z - \mathfrak{E}_z\mathfrak{H}_y) \qquad +\frac{c}{4\pi}\left(\mathfrak{E}_y\frac{\partial \mathfrak{H}_z}{\partial x} - \mathfrak{E}_z\frac{\partial \mathfrak{H}_y}{\partial x}\right)$$

-------------------- --------------------

-------------------- --------------------

S_x S_y S_z

[p. 75] Vector of the energy flow $\frac{c}{4\pi}(\mathfrak{E}_y\mathfrak{H}_z - \mathfrak{E}_z\mathfrak{H}_y)$

$$\frac{dE}{dt} + \text{heat loss} = \int S_n \cdot d\sigma.$$

Thus, the energy principle has been satisfied, with the expression for the energy being the same as in electrostatics.
The law of conservation of momentum; radiation pressure.

$$m_\nu \frac{d^2x_\nu}{dt} = X_\nu$$

Law of the equality of action & reaction $\sum X_n = 0$

From this $\sum m_1 \frac{d^2x_1}{dt^2} = 0$ for complete system. $\frac{d}{dt}\left[\sum m_1 \frac{dx_1}{dt}\right] = 0$ $\sum m_1 \frac{dx_1}{dt} = \text{const.}$

If external forces $\frac{d}{dt}\sum m_1 \frac{dx_1}{dt} = \sum X_a$

Can the momentum of a system be increased by internal electrom. processes (Can a system start moving by itself?) We must calculate the sum of the ponderomotive forces acting on the system. Per unit volume

$$j_y\mathfrak{H}_z - j_z\mathfrak{H}_y = -\frac{1}{4\pi c}\left\{\frac{\partial \mathfrak{E}_y}{\partial t}\mathfrak{H}_z - \frac{\partial \mathfrak{E}_z}{\partial t}\mathfrak{H}_y\right\} - \frac{1}{8\pi}\frac{\partial}{\partial x}(\mathfrak{H}_y^2 + \mathfrak{H}_z^2)$$

$$+\frac{1}{4\pi}\left\{\frac{\partial}{\partial z}(\mathfrak{H}_x\mathfrak{H}_z) + \frac{\partial}{\partial y}(\mathfrak{H}_x\mathfrak{H}_y)\right\}\left(-\frac{1}{4\pi}\mathfrak{H}_x\underbrace{\left(\frac{\partial \mathfrak{H}_y}{\partial y} + \frac{\partial \mathfrak{H}_z}{\partial z}\right)}_{-\frac{\partial \mathfrak{H}_x}{\partial x}}\right) + \frac{1}{8\pi}\frac{\partial \mathfrak{H}_x^2}{\partial x}$$

$$0 = \frac{1}{4\pi c}\left\{\frac{\partial \mathfrak{H}_y}{\partial t}\mathfrak{E}_z - \frac{\partial \mathfrak{H}_z}{\partial t}\mathfrak{E}_y\right\} - \frac{1}{8\pi}\frac{\partial}{\partial x}(\mathfrak{E}_y^2 + \mathfrak{E}_z^2) + \frac{1}{4\pi}\{\text{-------}\}$$

$$-\mathfrak{E}_x\rho + \frac{1}{8\pi}\frac{\partial(\mathfrak{E}_x^2)}{\partial x}$$

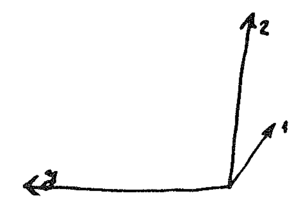
$\mathfrak{E}_x\rho + j_y\mathfrak{H}_z - j_z\mathfrak{H}_y$	$\frac{\partial}{\partial x}\left(-\frac{1}{2}\mathfrak{H}^2 + \mathfrak{H}_x^2\right) + \frac{\partial}{\partial y}(\mathfrak{H}_x\mathfrak{H}_y) + \frac{\partial}{\partial z}(\mathfrak{H}_x\mathfrak{H}_z)$
$= -\frac{\partial}{\partial t}\left\{\frac{1}{4\pi c}(\mathfrak{E}_y\mathfrak{H}_z - \mathfrak{E}_z\mathfrak{H}_y)\right\}$	$+\frac{\partial}{\partial x}\left(-\frac{1}{2}\mathfrak{E}^2 + \mathfrak{E}_x^2\right) + \frac{\partial}{\partial y}(\mathfrak{E}_x\mathfrak{E}_y) + \frac{\partial}{\partial z}(\mathfrak{E}_x\mathfrak{E}_z)$
$+\frac{1}{4\pi}$	

integrated over the whole system [p. 76]

$$\frac{d\mathfrak{J}_x}{dt} = -\frac{\partial}{\partial t}\underbrace{\int \frac{1}{4\pi c}(\mathfrak{E}_y\mathfrak{H}_z - \mathfrak{E}_z\mathfrak{H}_y)}_{J_x}$$

$$\mathfrak{J}_x + \mathfrak{J}_x = \text{konst.}$$

If we call $\mathfrak{J}_x$ the momentum of the electromagnetic field, then this tells us that the sum of the mechanical and the electromagnetic momentum in a complete system is constant.
Application to a plane wave ‖ x-axis
Electric force in Y-direction.



$$\mathfrak{E}_y = \mathfrak{H}_z \qquad \mathfrak{J}_{[-]=1} = \frac{1}{8\pi c}(\mathfrak{E}_y^2 + \mathfrak{H}_z^2) = \frac{1}{c}E_1$$

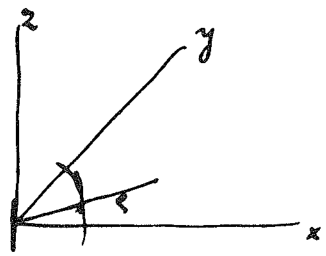
Momentum that impinges on the surface l per unit time = $\frac{I}{c}$ This is equal to radiation pressure.

Terms that were canceled out in integration. Maxwell stresses. The momentum transmitted by them to a unit of volume per unit time = momentum transm. to mech. system + increase of momentum in the element.

$$\frac{1}{c}\frac{\partial \mathfrak{E}}{\partial t} = \text{curl}\,\mathfrak{H} \quad -\frac{1}{c}\frac{\partial \mathfrak{H}}{\partial t} = \text{curl}\,\mathfrak{E} \quad \frac{\partial \mathfrak{H}_x}{\partial z} - \frac{\partial \mathfrak{H}_z}{\partial x}$$

[p. 77]

Hertz's Oscillator[76]



$$\mathfrak{H}_x = \frac{\partial\psi}{\partial y} \quad\Big|\quad \frac{\partial\psi}{\partial\rho}\cdot\frac{y}{\rho} \quad\Big|\quad x$$

$$\mathfrak{H}_y = -\frac{\partial\psi}{\partial x} \quad\Big|\quad \frac{\partial\psi}{\partial\rho}\cdot -\frac{x}{\rho} \quad\Big|\quad y \Bigg\} 0$$

$$\left\langle\frac{1}{c}\right\rangle \frac{\partial\mathfrak{E}_x}{\partial t} = \frac{\partial^2\psi}{\partial x\partial z} = \frac{\partial}{\partial t}\frac{\partial^2 F}{\partial x\partial z} \qquad \frac{\partial F}{\partial t} = c\psi$$

$$\left\langle\frac{1}{c}\right\rangle \frac{\partial\mathfrak{E}_y}{\partial t} = \frac{\partial^2\psi}{\partial y\partial z} = \text{-----}$$

$$\left\langle\frac{1}{c}\right\rangle \frac{\partial\mathfrak{E}_z}{\partial t} = -\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} = \text{----}$$ [77]

$$\mathfrak{E}_x = \frac{\partial^2 F}{\partial x\partial z} \qquad \mathfrak{H}_x = \frac{1}{c}\frac{\partial^2 F}{\partial y\partial t}$$

$$\mathfrak{E}_y = \frac{\partial^2 F}{\partial y\partial z} \qquad \mathfrak{H}_y = -\frac{1}{c}\frac{\partial^2 F}{\partial x\partial t}$$

$$\mathfrak{E}_z = -\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial z^2}$$

$$= \frac{\partial^2 F}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$$

F depends only on r

$$\frac{\partial F}{\partial x} = \frac{dF}{dr}\cdot\frac{x}{r}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{d^2F}{dr^2}\frac{x^2}{r^2} + \frac{dF}{dr}\left(\frac{1}{r} - \frac{x^2}{r^2}\right)$$ [78]

$$\Delta F = \frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right)$$

Here

$$\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = \Delta F$$

$$\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = \frac{1}{r^2}\frac{d}{dt}\left(r^2\frac{dF}{dr}\right)$$

Solution $F = \frac{1}{r} f\left(t - \frac{r}{2}\right)$

$$\frac{\partial F}{\partial r} = -\frac{1}{r^2}f - \frac{1}{cr}\dot f \;\Big|\; r^2\frac{\partial F}{\partial r} = -f - \frac{1}{c}$$

$$\frac{d}{dr}(\;\;) = \frac{1}{c}\dot f - \frac{1}{c}\dot f + \frac{1}{c^2} r\ddot f$$

$$\frac{1}{c^2}\cdot\frac{1}{r}\frac{d^2 f}{dt^2} = \frac{1}{r^2}\cdot\frac{1}{c^2} r\ddot f \quad \text{q.e.d.}$$

Behavior in immediate vicinity of the oscillator.

$$\mathfrak{E} = -\frac{\partial\varphi}{\partial z} \qquad \varphi = -\frac{\partial\frac{f}{r}}{\partial z} \text{ Pot. of a dipole}$$

$$e \begin{matrix} +\varepsilon \\ -\varepsilon \end{matrix} \qquad \frac{\varepsilon}{r} - \frac{\varepsilon}{r'} \qquad \frac{\partial\frac{\varepsilon l}{r}}{\partial z} \qquad \frac{1}{r'} = \frac{1}{r} - \frac{\partial\frac{1}{r}}{\partial z} l$$

f is the moment of the dipole $\varepsilon l = f$

Thus, process oscillation of dipole, which is ∞ small compared with the wave length. Calculation of the energy radiated outwards.[79]

[p. 78]

$$\mathfrak{E}_x = \frac{xz}{c^2r^3}\ddot f \qquad \mathfrak{H}_x = -\frac{y}{c^2r^3}\ddot f \qquad z^2x^2$$

$$\mathfrak{E}_y = \frac{yz}{c^2r^3}\ddot f \qquad \mathfrak{H}_y = \frac{x}{c^2r^3}\ddot f \qquad z^2y^2$$

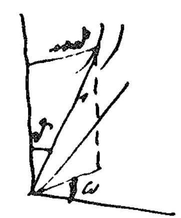
$$z^4 - 2z^2r^2 + r^4$$

$$\mathfrak{E}_z = -\frac{x^2+y^2}{c^2r^3}\ddot f \qquad \mathfrak{H}_z = 0$$

$$r^4 - z^2r^2 \qquad \sqrt{} = r^2\left(1 - \frac{z^2}{r^2}\right) =$$

$$\mathfrak{E} \perp \mathfrak{H} \qquad \mathfrak{E}\ \&\ \mathfrak{H} \perp xyz \qquad |\mathfrak{E}| = \frac{1}{c^2r}\ddot f \sin^2\vartheta = |\mathfrak{H}|$$ [80]

Rad[iation of?] energy[81]



$$\int_T\int |\mathfrak{E}||\mathfrak{H}| r^2\, d\Omega = \frac{1}{c^4}\int \ddot f^2\, dt \underbrace{\int \frac{\sin^2\vartheta\, dw}{2\pi\sin\vartheta\, d\vartheta}}_{2\pi\int_0^\pi (1-\cos^2\vartheta)\sin\vartheta\, d\vartheta}$$

$$= \left|-\cos\vartheta + \frac{\cos^3\vartheta}{3}\right|_0^\pi = 2 - \frac{2}{3} = \frac{4}{3}$$

In unit time $\frac{c}{4\pi}\cdot\frac{4\cdot 2\pi\overline{\ddot f^2}}{3c^4} = \frac{2}{3c^3}\overline{\ddot f^2}$

If excitory sines[82] $f = f_0\cos(2\pi nt)$, then $\ddot f = f_0(2\pi n)^2\cos(\;)$

$$\overline{\ddot f^2} = \frac{1}{2}f_0^2(2\pi n)^4$$

$$A = \frac{1}{3c^3}(2\pi n)^4 f_0^2$$

$$l = 100 \qquad p_{max} = 3 \qquad C = 30 \qquad n = 10^8$$

$$f_0 = 10^4 \qquad 2\pi n = 6\ 10^8$$

$$A = \frac{10^{35}\ 10^8}{3\cdot 27\cdot 10^{30}} \approx 10^{11} = 2000 \text{ cal. per sec.}$$